Respuesta Dinámica Exacta de Modelo LTI en DT Continuo $t[s] \in \mathbb{R}$: Solución Analítica de la Ecuación Diferencial de Estado LTI

$$\boldsymbol{S}_{LTI}\begin{cases}\dot{\boldsymbol{x}}(t) = \boldsymbol{A}.\boldsymbol{x}(t) + \boldsymbol{B}.\boldsymbol{u}(t); \ \ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) = \boldsymbol{C}.\boldsymbol{x}(t) + \boldsymbol{D}.\boldsymbol{u}(t)\end{cases}$$; Tiempo Continuo $t[s] \in \mathbb{R}$

A) Respuesta **Natural** (causada solo por **Estado Inicial** NO Nulo, SIN Excitación posterior):
Solución de la *Ecuación Diferencial de Estado* ***Autónoma u Homogénea***

$\underline{\boldsymbol{u}(t) \equiv \boldsymbol{0}}$ $\therefore$ $\boxed{\dot{\boldsymbol{x}}(t) = \boldsymbol{A}.\boldsymbol{x}(t)}$; $\underline{\boldsymbol{x}(t_0) = \boldsymbol{x}_0}$; $\boldsymbol{y}(t) = \boldsymbol{C}.\boldsymbol{x}(t)$

1. **Solución en DT**: Proponemos solución general $\boldsymbol{x}(t)$ = serie de potencias de $(t - t_0)$:

$$\boldsymbol{x}(t) = \boldsymbol{b}_0 + \boldsymbol{b}_1.(t-t_0) + \boldsymbol{b}_2.(t-t_0)^2 + \boldsymbol{b}_3.(t-t_0)^3 + \cdots + \boldsymbol{b}_k.(t-t_0)^k + \cdots = \sum_{k=0}^{\infty} \boldsymbol{b}_k.(t-t_0)^k$$

Evaluamos $\boldsymbol{x}(t)$ en instante inicial $t = t_0$: $\boldsymbol{x}(t_0) = \boldsymbol{b}_0 = \boldsymbol{x}_0 = \boldsymbol{I}.\boldsymbol{x}_0$

a) Derivamos $\boldsymbol{x}(t)$ con respecto al tiempo:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{b}_1 + 2.\boldsymbol{b}_2.(t-t_0) + 3.\boldsymbol{b}_3.(t-t_0)^2 + \cdots + k.\boldsymbol{b}_k.(t-t_0)^{k-1} + \cdots = \sum_{k=1}^{\infty} k.\boldsymbol{b}_k.(t-t_0)^{k-1}$$

b) Reemplazamos $\boldsymbol{x}(t)$ propuesta en la Ecuación Diferencial de Estado Autónoma original:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}.\boldsymbol{b}_0 + \boldsymbol{A}.\boldsymbol{b}_1.(t-t_0) + \boldsymbol{A}.\boldsymbol{b}_2.(t-t_0)^2 + \cdots + \boldsymbol{A}.\boldsymbol{b}_k.(t-t_0)^k + \cdots = \sum_{k=0}^{\infty} \boldsymbol{A}.\boldsymbol{b}_k.(t-t_0)^k$$

Comparamos a) y b) → igualamos coeficientes de cada término de igual potencia de $(t - t_0)$:

$\boldsymbol{b}_1 = \boldsymbol{A}.\boldsymbol{b}_0 \ \therefore \ \boldsymbol{b}_1 = \boldsymbol{A}.\boldsymbol{x}_0$; $2.\boldsymbol{b}_2 = \boldsymbol{A}.\boldsymbol{b}_1 \ \therefore \ \boldsymbol{b}_2 = \frac{\boldsymbol{A}^2.\boldsymbol{x}_0}{2}$; $3.\boldsymbol{b}_3 = \boldsymbol{A}.\boldsymbol{b}_2 \ \therefore \ \boldsymbol{b}_3 = \frac{\boldsymbol{A}^3.\boldsymbol{x}_0}{3.2}$; ...

$k.\boldsymbol{b}_k = \boldsymbol{A}.\boldsymbol{b}_{k-1} \ \therefore \ \boldsymbol{b}_k = \dfrac{\boldsymbol{A}^k.\boldsymbol{x}_0}{k!}$

Reemplazamos coeficientes en la solución general $\boldsymbol{x}(t)$ propuesta originalmente:

$$\boldsymbol{x}(t) = \boldsymbol{I}.\boldsymbol{x}_0 + \boldsymbol{A}.\boldsymbol{x}_0.(t-t_0) + \frac{\boldsymbol{A}^2.\boldsymbol{x}_0}{2!}.(t-t_0)^2 + \cdots + \frac{\boldsymbol{A}^k.\boldsymbol{x}_0}{k!}.(t-t_0)^k + \cdots = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k.\boldsymbol{x}_0}{k!}.(t-t_0)^k$$

$$\boldsymbol{x}(t) = \left[\boldsymbol{I} + \boldsymbol{A}.(t-t_0) + \frac{\boldsymbol{A}^2.(t-t_0)^2}{2!} + \cdots + \frac{\boldsymbol{A}^k.(t-t_0)^k}{k!} + \cdots\right].\boldsymbol{x}_0 = \left[\sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k.(t-t_0)^k}{k!}\right].\boldsymbol{x}_0$$

Función exponencial natural *escalar*: $e^{a.t} = 1 + a.t + \frac{a^2.t^2}{2!} + \cdots + \frac{a^k.t^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{a^k.t^k}{k!}$

Función Exponencial Matricial: $\boxed{\boldsymbol{\Phi}_{\boldsymbol{A}}(t) = e^{\boldsymbol{A}.t} \equiv \boldsymbol{I} + \boldsymbol{A}.t + \frac{\boldsymbol{A}^2.t^2}{2!} + \cdots + \frac{\boldsymbol{A}^k.t^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k.t^k}{k!}}$

Respuesta Natural: $\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}.(t-t_0)}.\boldsymbol{x}(t_0) = \boldsymbol{\Phi}_{\boldsymbol{A}}(t-t_0).\boldsymbol{x}(t_0)}$

$\boldsymbol{\Phi}_{\boldsymbol{A}}(t-t_0)$: **Matriz de Transición de Estado**, evolución de Estado inicial $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ → Estado actual $\boldsymbol{x}(t)$.

2. **Solución en DF**: aplicando Transformada de Laplace $\mathcal{L}$

DT: $t[s] \in \mathbb{R} \rightarrow$ **DF** $s[rad/s] = \sigma + j\omega \in \mathbb{C}$

$\mathcal{L}\{\dot{\boldsymbol{x}}(t) = \boldsymbol{A}.\boldsymbol{x}(t)\}$

$s.\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}.\boldsymbol{X}(s)$

$[s.\boldsymbol{I} - \boldsymbol{A}].\boldsymbol{X}(s) = \boldsymbol{x}(0)$

$\boxed{\boldsymbol{X}(s) = [s.\boldsymbol{I} - \boldsymbol{A}]^{-1}.\boldsymbol{x}(0)}$, $[s.\boldsymbol{I} - \boldsymbol{A}]^{-1} = \mathcal{L}\{e^{\boldsymbol{A}.t}\}$

$\mathcal{L}^{-1}\{\boldsymbol{X}(s) = [s.\boldsymbol{I} - \boldsymbol{A}]^{-1}.\boldsymbol{x}(0)\}$

Respuesta Natural: $\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}.(t-0)}.\boldsymbol{x}(0) = \boldsymbol{\Phi}_{\boldsymbol{A}}(t-0).\boldsymbol{x}(0)}$

$$\boldsymbol{S_{LTI}}\begin{cases}\dot{\boldsymbol{x}}(t) = \boldsymbol{A}.\boldsymbol{x}(t) + \boldsymbol{B}.\boldsymbol{u}(t); \;\; \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) = \boldsymbol{C}.\boldsymbol{x}(t) + \boldsymbol{D}.\boldsymbol{u}(t)\end{cases}$$ **;** Tiempo Continuo $t[s] \in \mathbb{R}$

B) Respuesta **Forzada** (causada por **Excitación**) + Respuesta **Natural** (causada por Estado Inicial):
Solución de la *Ecuación Diferencial de Estado Completa,* ***Forzada o No Homogénea***

$$\boldsymbol{u}(t) \neq \boldsymbol{0} \;\therefore\; \boxed{\dot{\boldsymbol{x}}(t) = \boldsymbol{A}.\boldsymbol{x}(t) + \boldsymbol{B}.\boldsymbol{u}(t)}; \;\; \boldsymbol{x}(t_0) = \boldsymbol{x}_0; \;\; \boldsymbol{y}(t) = \boldsymbol{C}.\boldsymbol{x}(t) + \boldsymbol{D}.\boldsymbol{u}(t)$$

1. **Solución en DT**: Reordenamos y pre-multiplicamos por exponencial $e^{-\boldsymbol{A}.t}$:

$$e^{-\boldsymbol{A}.t}.[\dot{\boldsymbol{x}}(t) - \boldsymbol{A}.\boldsymbol{x}(t)] = e^{-\boldsymbol{A}.t}.\boldsymbol{B}.\boldsymbol{u}(t)$$

Por otro lado:

- $\frac{d}{dt}[e^{\boldsymbol{A}.t}] = \boldsymbol{A}.e^{\boldsymbol{A}.t} = e^{\boldsymbol{A}.t}.\boldsymbol{A}$ (Producto sí es Conmutativo, Propiedad de Exponencial Matricial)
- $\frac{d}{dt}[e^{-\boldsymbol{A}.t}.\boldsymbol{x}(t)] = e^{-\boldsymbol{A}.t}.\dot{\boldsymbol{x}}(t) - \boldsymbol{A}.e^{-\boldsymbol{A}.t}.\boldsymbol{x}(t) = e^{-\boldsymbol{A}.t}.[\dot{\boldsymbol{x}}(t) - \boldsymbol{A}.\boldsymbol{x}(t)]$

Reemplazamos en ecuación anterior:

$$\frac{d}{dt}[e^{-\boldsymbol{A}.t}.\boldsymbol{x}(t)] = e^{-\boldsymbol{A}.t}.\boldsymbol{B}.\boldsymbol{u}(t)$$

Integramos la ecuación con respecto a t en ambos miembros:

$$\int_{t_0}^{t}\frac{d}{d\tau}[e^{-\boldsymbol{A}.\tau}.\boldsymbol{x}(\tau)].d\tau = \int_{t_0}^{t} e^{-\boldsymbol{A}.\tau}.\boldsymbol{B}.\boldsymbol{u}(\tau).d\tau$$

$$[e^{-\boldsymbol{A}.\tau}.\boldsymbol{x}(\tau)]\Big|_{t_0}^{t} = e^{-\boldsymbol{A}.t}.\boldsymbol{x}(t) - e^{-\boldsymbol{A}.t_0}.\boldsymbol{x}_0 = \int_{t_0}^{t} e^{-\boldsymbol{A}.\tau}.\boldsymbol{B}.\boldsymbol{u}(\tau).d\tau$$

$$e^{-\boldsymbol{A}.t}.\boldsymbol{x}(t) = e^{-\boldsymbol{A}.t_0}.\boldsymbol{x}_0 + \int_{t_0}^{t} e^{-\boldsymbol{A}.\tau}.\boldsymbol{B}.\boldsymbol{u}(\tau).d\tau$$

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}.t}.e^{-\boldsymbol{A}.t_0}.\boldsymbol{x}_0 + e^{\boldsymbol{A}.t}.\int_{t_0}^{t} e^{-\boldsymbol{A}.\tau}.\boldsymbol{B}.\boldsymbol{u}(\tau).d\tau$$

Respuesta Natural + Respuesta Forzada: Integral de Convolución de Duhamel

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}.(t-t_0)}.\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A}.(t-\tau)}.\boldsymbol{B}.\boldsymbol{u}(\tau).d\tau} = e^{\boldsymbol{A}.(t-t_0)}.\boldsymbol{x}(t_0) + e^{\boldsymbol{A}.(t-t_0)} * \boldsymbol{B}.\boldsymbol{u}(t-t_0)$$

2. **Solución en DF**: aplicando Transformada de Laplace $\boldsymbol{\mathcal{L}}$

DT: $t[s] \in \mathbb{R} \rightarrow$ **DF** $s[rad/s] = \sigma + j\omega \in \mathbb{C}$

$$\boldsymbol{\mathcal{L}}\{\dot{\boldsymbol{x}}(t) = \boldsymbol{A}.\boldsymbol{x}(t) + \boldsymbol{B}.\boldsymbol{u}(t)\}$$

$$s.\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}.\boldsymbol{X}(s) + \boldsymbol{B}.\boldsymbol{U}(s)$$

$$[s.\boldsymbol{I} - \boldsymbol{A}].\boldsymbol{X}(s) = \boldsymbol{x}(0) + \boldsymbol{B}.\boldsymbol{U}(s)$$

$$\boxed{\boldsymbol{X}(s) = [s.\boldsymbol{I} - \boldsymbol{A}]^{-1}.\boldsymbol{x}(0) + [s.\boldsymbol{I} - \boldsymbol{A}]^{-1}.\boldsymbol{B}.\boldsymbol{U}(s)}, \; [s.\boldsymbol{I} - \boldsymbol{A}]^{-1} = \boldsymbol{\mathcal{L}}\{e^{\boldsymbol{A}.t}\}$$

$$\boldsymbol{\mathcal{L}}^{-1}\{\boldsymbol{X}(s) = [s.\boldsymbol{I} - \boldsymbol{A}]^{-1}.\boldsymbol{x}(0) + [s.\boldsymbol{I} - \boldsymbol{A}]^{-1}.\boldsymbol{B}.\boldsymbol{U}(s)\}$$

Respuesta Natural + Respuesta Forzada:

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}.(t-0)}.\boldsymbol{x}(0) + e^{\boldsymbol{A}.(t-0)} * \boldsymbol{B}.\boldsymbol{u}(t-0)}$$

Respuesta Dinámica Aproximada de Modelo LTI en DT Discretizado con MUESTREO Periódico $t_k[\mathrm{s}] \equiv k.T_s \in \mathbb{R}, k \in \mathbb{Z}$ y excitación c/ ZOH

$$\boldsymbol{S_{LTI}} \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}.\boldsymbol{x}(t) + \boldsymbol{B}.\boldsymbol{u}(t); \;\; \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) = \boldsymbol{C}.\boldsymbol{x}(t) + \boldsymbol{D}.\boldsymbol{u}(t) \end{cases}$$

; Tiempo Continuo $t[\mathrm{s}] \in \mathbb{R}$

Respuesta Natural + Respuesta Forzada: Integral de Convolución de Duhamel

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}.(t-t_0)}.\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A}.(t-\tau)}.\boldsymbol{B}.\boldsymbol{u}(\tau).d\tau$$

Muestreo Periódico: $t_k[\mathrm{s}] \equiv k.T_s \in \mathbb{R} \rightarrow$ DT Discreto $k \in \mathbb{Z}$

($T_s[\mathrm{s}] \in \mathbb{R}$: Periodo de Muestreo)

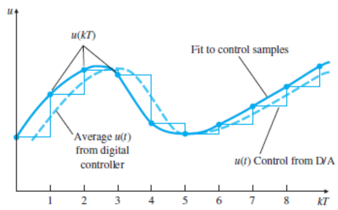
Evaluamos Respuesta exacta entre instantes correspondientes a dos muestras consecutivas de tiempo:

$t_0 \equiv t_k \equiv k.T_s; \;\; t \equiv t_{k+1} \equiv (k+1).T_s \;\; \rightarrow \;\; t - t_0 = T_s$

$$\boldsymbol{x}\big((k+1).T_s\big) = e^{\boldsymbol{A}.T_s}.\boldsymbol{x}(k.T_s) + \int_{k.T_s}^{(k+1).T_s} e^{\boldsymbol{A}.((k+1).T_s-\tau)}.\boldsymbol{B}.\boldsymbol{u}(\tau).d\tau$$

Aproximamos $\boldsymbol{u}(\tau) \cong \boldsymbol{u}(k.T_s)$ **"Constante"** en intervalo: $\forall\, \tau \in [k.T_s, (k+1).T_s)$ → fuera de la integral:

Nota: esto equivale a considerar la salida de un **retenedor de orden 0 (ZOH, Zero–Order Hold)** como entrada $\boldsymbol{u}(t)$, válido para la acción de control desde un controlador digital a través de un Convertidor D/A (retardo $\sim T_s/2$).



$$\boldsymbol{x}\big((k+1).T_s\big) \cong e^{\boldsymbol{A}.T_s}.\boldsymbol{x}(k.T_s) + \left[\int_{k.T_s}^{(k+1).T_s} e^{\boldsymbol{A}.((k+1).T_s-\tau)}.d\tau\right].\boldsymbol{B}.\boldsymbol{u}(k.T_s)$$

Resolvemos la integral definida del 2° término por sustitución de variables:

$\eta \equiv (k+1).T_s - \tau \;\; \therefore \tau = (k+1).T_s - \eta; \;\; d\tau = -d\eta;$

$\tau = k.T_s \;\; \Rightarrow \;\; \eta = T_s;$

$\tau = (k+1).T_s \;\; \Rightarrow \;\; \eta = 0$

$$\boxed{\boldsymbol{x}\big((k+1).T_s\big) \cong e^{\boldsymbol{A}.T_s}.\boldsymbol{x}(k.T_s) + \left[\int_{0}^{T_s} e^{\boldsymbol{A}.\eta}.d\eta\right].\boldsymbol{B}.\boldsymbol{u}(k.T_s)}$$

Ecuación en Diferencias de Estado en DT Discretizado $t_k[\mathrm{s}] \equiv k.T_s \in \mathbb{R} \rightarrow$ DT Discreto $k \in \mathbb{Z}$

$\boxed{\boldsymbol{x}\big((k+1).T_s\big) \cong \boldsymbol{\Phi_A}(T_s).\boldsymbol{x}(k.T_s) + \boldsymbol{\Gamma_{A,B}}(T_s).\boldsymbol{u}(k.T_s)}$: Ecuación en Diferencias de Estado

$\boldsymbol{y}(k.T_s) = \boldsymbol{C}.\boldsymbol{x}(k.T_s) + \boldsymbol{D}.\boldsymbol{u}(k.T_s)$: Ecuación Algebraica de Salida

Cálculo de Matrices de Sistema (o de Transición) $\mathbf{\Phi}_{\boldsymbol{A}}(T_s)$ y de Entrada $\mathbf{\Gamma}_{\boldsymbol{A},\boldsymbol{B}}(T_s)$ en DT Discretizado:

$$\boxed{\mathbf{\Phi}_{\boldsymbol{A}}(T_s) \equiv e^{\boldsymbol{A}.T_s}} = \boldsymbol{I} + \boldsymbol{A}.T_s + \frac{\boldsymbol{A}^2.T_s{}^2}{2!} + \cdots + \frac{\boldsymbol{A}^k.T_s{}^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k.T_s{}^k}{k!}$$

$$= \boldsymbol{I} + \boldsymbol{A}.T_s.\left[\boldsymbol{I} + \frac{\boldsymbol{A}.T_s}{2!} + \frac{\boldsymbol{A}^2.T_s{}^2}{3!} + \cdots + \frac{\boldsymbol{A}^{k-1}.T_s{}^{k-1}}{k!} + \cdots\right] = \boldsymbol{I} + \boldsymbol{A}.T_s.\left[\sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k.T_s{}^k}{(k+1)!}\right]$$

$$\boxed{\mathbf{\Phi}_{\boldsymbol{A}}(T_s) \equiv e^{\boldsymbol{A}.T_s} = \boldsymbol{I} + \boldsymbol{A}.T_s.\left[\sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k.T_s{}^k}{(k+1)!}\right] = \boldsymbol{I} + \boldsymbol{A}.T_s.\mathbf{\Psi}_{\boldsymbol{A}}(T_s)}$$

$$\boxed{\mathbf{\Gamma}_{\boldsymbol{A},\boldsymbol{B}}(T_s) \equiv \left[\int_0^{T_s} e^{\boldsymbol{A}.\eta}.d\eta\right].\boldsymbol{B}} = \left[\int_0^{T_s}\left(\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^k.\eta^k}{k!}\right).d\eta\right].\boldsymbol{B} = \left[\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^k}{k!}.\int_0^{T_s}\eta^k.d\eta\right].\boldsymbol{B}$$

$$= \left[\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^k}{k!}.\left(\frac{\eta^{k+1}}{k+1}\right)\Bigg|_0^{T_s}\right].\boldsymbol{B} = \left[\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^k}{k!}.\frac{T_s{}^{k+1}}{k+1}\right].\boldsymbol{B} = \left[\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^k.T_s{}^k}{(k+1)!}\right].T_s.\boldsymbol{B}$$

$$\boxed{\mathbf{\Gamma}_{\boldsymbol{A},\boldsymbol{B}}(T_s) \equiv \left[\int_0^{T_s} e^{\boldsymbol{A}.\eta}.d\eta\right].\boldsymbol{B} = \left[\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^k.T_s{}^k}{(k+1)!}\right].T_s.\boldsymbol{B} = \mathbf{\Psi}_{\boldsymbol{A}}(T_s).T_s.\boldsymbol{B}}$$

Caso **particular**: $\exists \boldsymbol{A}^{-1}, \boldsymbol{A}$ no singular o invertible:

$$\mathbf{\Gamma}_{\boldsymbol{A},\boldsymbol{B}}(T_s) \equiv \left[\int_0^{T_s} e^{\boldsymbol{A}.\eta}.d\eta\right].\boldsymbol{B} = \left[\boldsymbol{A}^{-1}.e^{\boldsymbol{A}.\eta}\Big|_0^{T_s}\right].\boldsymbol{B} = \boldsymbol{A}^{-1}.\left(e^{\boldsymbol{A}.T_s} - \boldsymbol{I}\right).\boldsymbol{B} = \left[\boldsymbol{A}^{-1}.\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^{k+1}.T_s{}^{k+1}}{(k+1)!}\right].\boldsymbol{B}$$

Definimos $\mathbf{\Psi}_{\boldsymbol{A}}(T_s)$ como:

$$\mathbf{\Psi}_{\boldsymbol{A}}(T_s) \equiv \left[\boldsymbol{I} + \frac{\boldsymbol{A}.T_s}{2!} + \frac{\boldsymbol{A}^2.T_s{}^2}{3!} + \cdots + \frac{\boldsymbol{A}^{k-1}.T_s{}^{k-1}}{k!} + \frac{\boldsymbol{A}^k.T_s{}^k}{(k+1)!} + \cdots\right] = \left[\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^k.T_s{}^k}{(k+1)!}\right]$$

$$\cong \boldsymbol{I} + \frac{\boldsymbol{A}.T_s}{2}.\left\{\boldsymbol{I} + \frac{\boldsymbol{A}.T_s}{3}.\left[\boldsymbol{I} + \frac{\boldsymbol{A}.T_s}{4}.\left(\boldsymbol{I} + \cdots\frac{\boldsymbol{A}.T_s}{N-1}.\left(\boldsymbol{I} + \frac{\boldsymbol{A}.T_s}{N}\right)\right)\right]\right\}$$

Aproximación recursiva hasta factor N, tiene **mejores propiedades numéricas** [ver Papers Moler and van Loan, 1978, 2003].

→ Calculamos:

$$\mathbf{\Phi}_{\boldsymbol{A}}(T_s) = \boldsymbol{I} + \boldsymbol{A}.T_s.\mathbf{\Psi}_{\boldsymbol{A}}(T_s)$$
$$\mathbf{\Gamma}_{\boldsymbol{A},\boldsymbol{B}}(T_s) = \mathbf{\Psi}_{\boldsymbol{A}}(T_s).T_s.\boldsymbol{B}$$

IMPORTANTE: **Función c2d** en Matlab, Control Systems Toolbox

En la práctica, la evaluación de estas matrices $\mathbf{\Phi}_{\boldsymbol{A}}(T_s)$ y $\mathbf{\Gamma}_{\boldsymbol{A},\boldsymbol{B}}(T_s)$ se puede realizar ***numéricamente*** a partir de las matrices $\boldsymbol{A}, \boldsymbol{B}$ y el Periodo de Muestreo T_s mediante la **función c2d** en Matlab, Control Systems Toolbox: sysd = c2d(sysc,Ts).