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ME8281 Advanced Control Systems Design

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(1/19/2006)

1 Jacobian Linearization

Consider a nonlinear differential system,

$$\dot{x} = f(x, t, u) \tag{1}$$

where $x \in \Re^n$, $u \in \Re^m$. i.e. $f(x, t, u) \in \Re^n$.

Suppose that given a nominal input $\bar{u}(t)$, there is a nominal state trajectory denoted by $\bar{x}(t)$. By this, it is meant that $\bar{x}(t)$ and $\bar{u}(t)$ satisfy, for all t in the relevant time period $[t_0, t_1]$

$$\dot{\bar{x}}(t) = f(\bar{x}(t), t, \bar{u}(t)) \tag{2}$$

In many cases, we are interested in the steady operating condition. In this case, $\bar{x}(t) = x_e$ is a constant, called the equilibrium state, in the sense that:

$$\dot{x}_e = f(x_e, t, \bar{u}(t)) = 0$$

We are interested in finding out what happens when either the initial state or inputs deviate from the nominal initial state or inputs,

$$x(t_0) = \bar{x}(t_0) + \delta x(t_0) \tag{3}$$
$$u(t) = u(t) + \delta u(t) \tag{4}$$

While one can plug the deviated $x(t_0)$ and $u(t)$ into (1) to find out what happens, it is cumbersome and more importantly, does not provide an easy means of analyzing and designing controllers. Instead, we will obtain a *linear* perturbation model to approximately describe $\delta x(t)$. Linear model is simpler, easier to analyze and provides more insights.

Write the perturbed state as

$$x(t) = \bar{x}(t) + \delta x(t).$$

Then,

$$\dot{x}(t) = \dot{\bar{x}}(t) + \frac{d}{dt}\delta x(t)$$

Hence,

$$\begin{aligned}\frac{d}{dt}\delta x(t) &= f(x(t), t, u(t)) - f(\bar{x}(t), t, \bar{u}(t)) \\ &= f(\bar{x}(t) + \delta x(t), t, \bar{u}(t) + \delta u(t)) - f(\bar{x}(t), t, \bar{u}(t))\end{aligned}$$

Now using Taylor series expansion for each row $(i = 1, \ldots, n)$ of the vector function f, we have:

$$\begin{aligned}&= f_i(\bar{x}(t) + \delta x(t), t, \bar{u}(t)) \\ &= f_i(\bar{x}(t), t, \bar{u}(t)) + \sum_{j=1}^{n} \left.\frac{\partial f_i}{\partial x_j}\right|_{\bar{x}(t), t, \bar{u}(t)} \delta x_j(t) + \sum_{k=1}^{m} \left.\frac{\partial f_i}{\partial u_k}\right|_{\bar{x}(t), t, \bar{u}(t)} \delta u_k(t) + \text{H.O.T.}\end{aligned}$$

such that the higher order terms vanish as $\|\delta x\|$ and $\|\delta u\|$ vanish. Thus,

$$\frac{d}{dt}\delta x(t) \approx A(t)\delta x(t) + B(t)\delta u(t) \tag{5}$$

where $A(t) \in \Re^{n\times n}$ and $B(t) \in \Re^{n\times m}$,

$$A_{ij}(t) = \left.\frac{\partial f_i}{\partial x_j}\right|_{\bar{x}(t),t,\bar{u}(t)} \qquad B_{ik}(t) = \left.\frac{\partial f_i}{\partial u_k}\right|_{\bar{x}(t),t,\bar{u}(t)} \tag{6}$$

Notice that even if (1) is time-invariant, the linearized system about a non-equilibrium nominal state is typically a time varying system.

Eq. (5) is the Jacobian linearized approximation of the perturbations. Assuming (we will develop these tools) that we can analyze or obtain solution to linear differential systems like Eq. (5), we can constitute $x(t)$ from $x(t) = \bar{x}(t) + \delta x(t)$.