# State estimation

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Control y Sistemas

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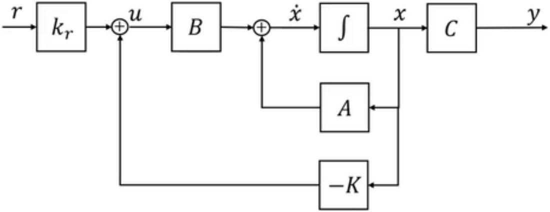
# State feedback control

***Idea with state feedback control design*:** Modify the eigenvalues of the system by using the input, $u = -Kx + k_r r$.




***Problem:*** Requires full access to the state vector, $u = -Kx$

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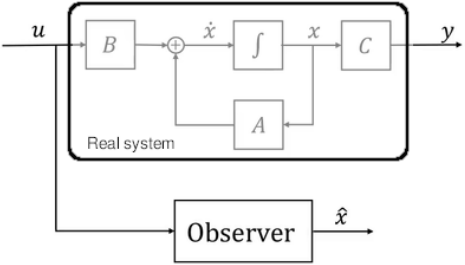
***Idea of state estimation***: Develop an observer of the dynamic system that provides an estimate, $\hat{x}$, of the system's states.



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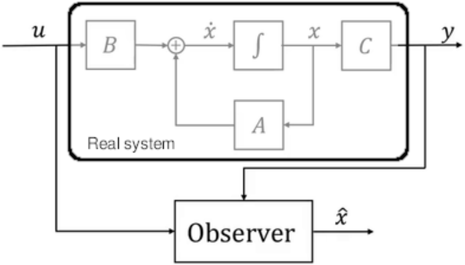

# State estimation – Open loop estimator

***Goal of state estimation***: $\tilde{x} = x - \hat{x} = 0$

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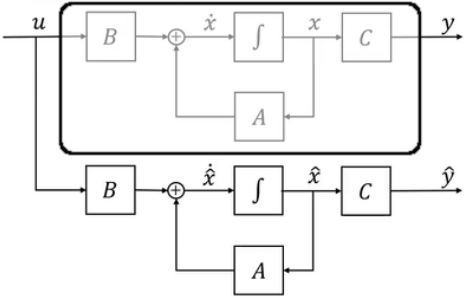

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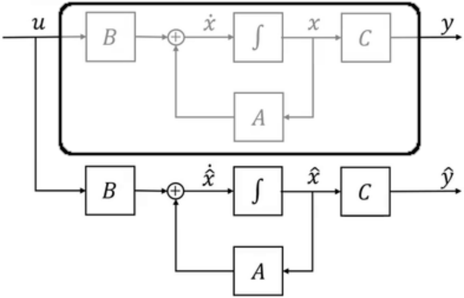


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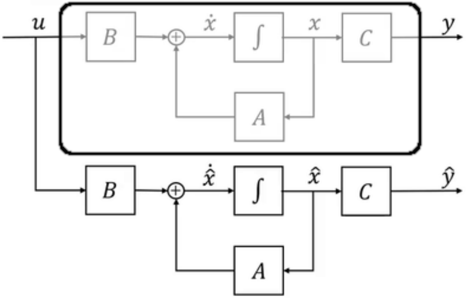


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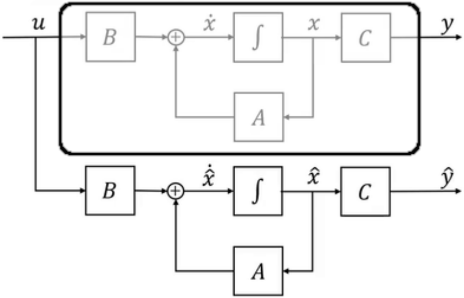


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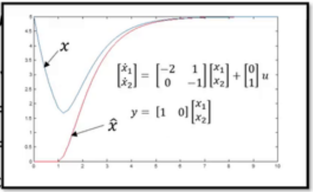


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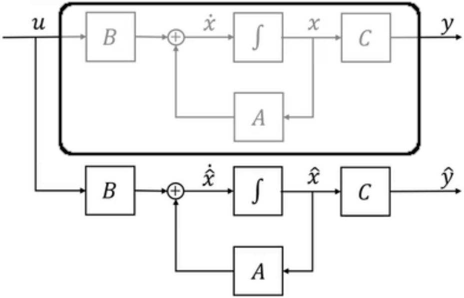


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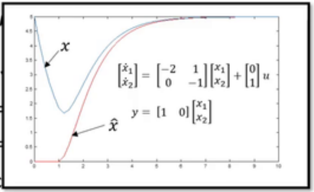


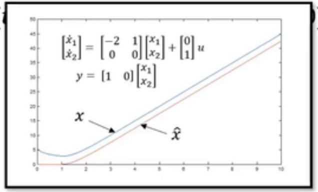

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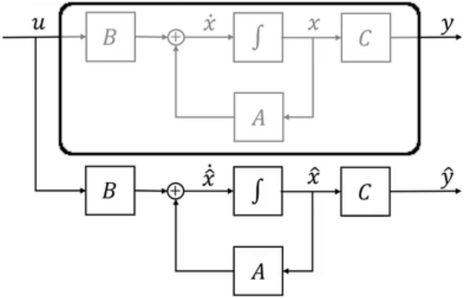


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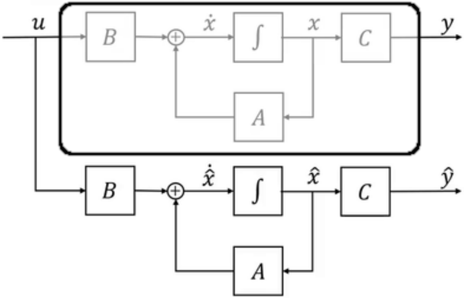


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*Open loop estimation does not seem to be a good idea!*

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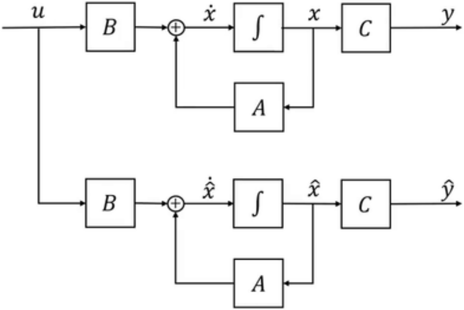

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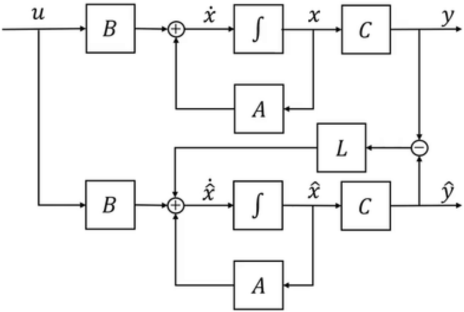


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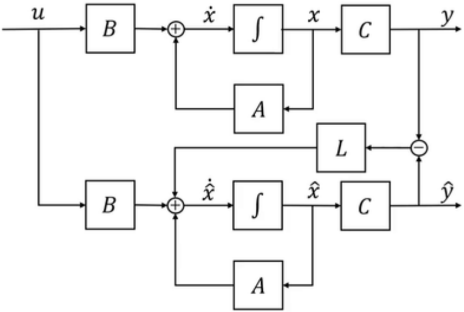


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Feed back the error to the open loop estimator via a feedback gain $L$:

$$\dot{\hat{x}} = A\hat{x} + Bu + L\tilde{y}$$

$$\hat{y} = C\hat{x}$$

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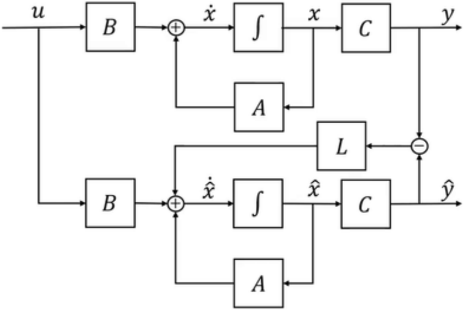


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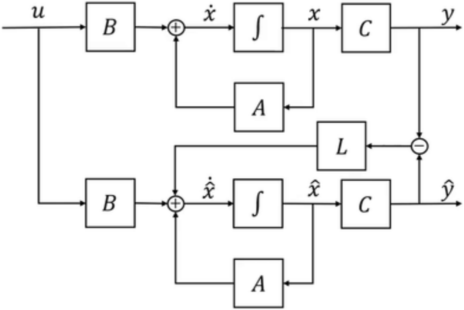


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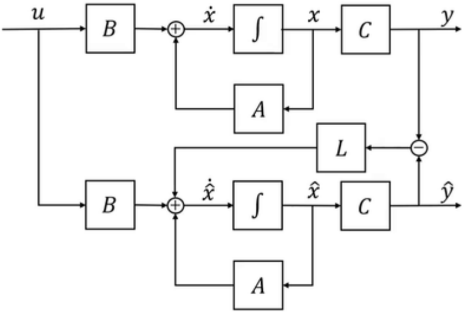


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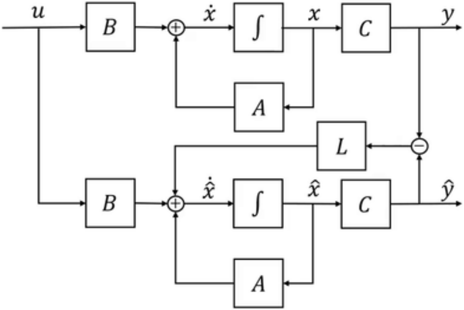


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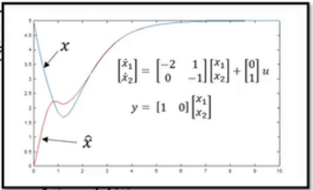


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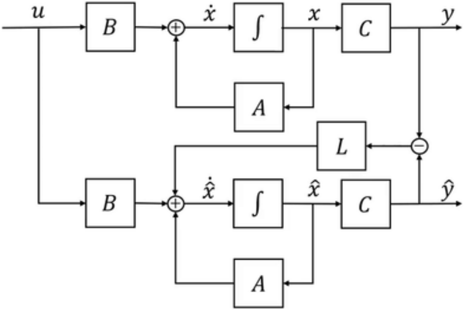


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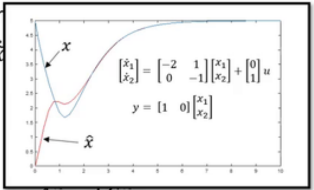


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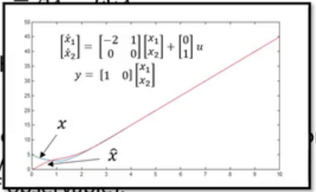


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Use pole placement with a desired characteristic polynomial to choose the estimator gain, $L$.

# Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 12\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}6\\ 3\end{bmatrix} u$$

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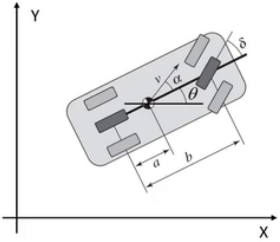
where $x_1$ is the lateral position Y , $x_2$ is the heading orientation $\theta$ and $u$ is the steering angle $\delta$.

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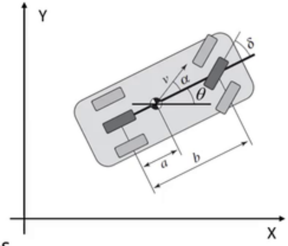


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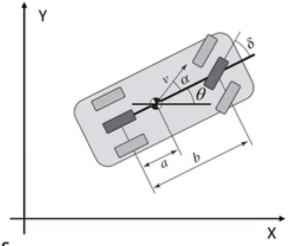


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***Specification:*** Desired characteristic polynomial:

$$p_{des}(\lambda) = (\lambda + 6)(\lambda + 4) = \lambda^2 + 10\lambda + 24$$

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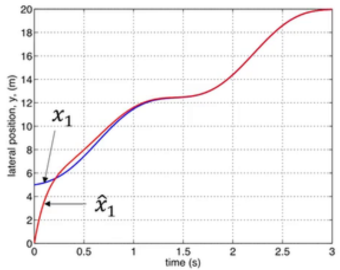

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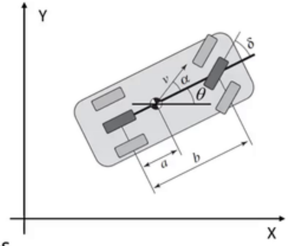


Design a state estimator to estimate the vehicle's states, from measurement of the heading angle.

Vehicle data: $v_0 = 12\ m/s$, $a = 2\ m$, $b = 4\ m$

***Specification:*** Desired characteristic polynomial:

$$p_{des}(\lambda) = (\lambda + 6)(\lambda + 4) = \lambda^2 + 10\lambda + 24$$

# Revisit Example - Vehicle steering (Ex 7.4)

State estimator:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The closed loop estimator dynamics becomes

$$\dot{\tilde{x}} = (A - LC)\tilde{x} = \begin{bmatrix} 0 & 12 - l_1 \\ 0 & -l_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

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It is not possible shape the error dynamics. We say that the system is not ***observable***.

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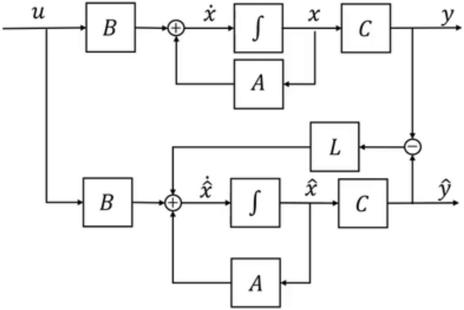
# Control using estimated states

Use the estimated states for feedback, $u = -K\hat{x} + k_r r$.



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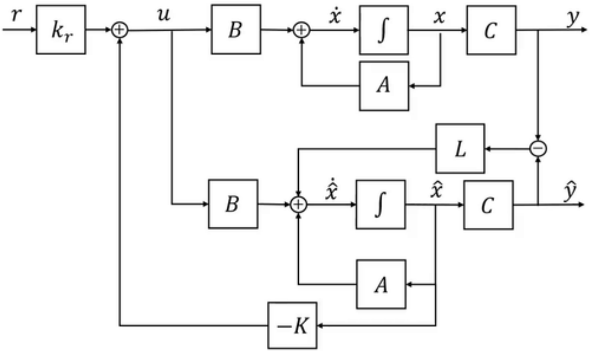

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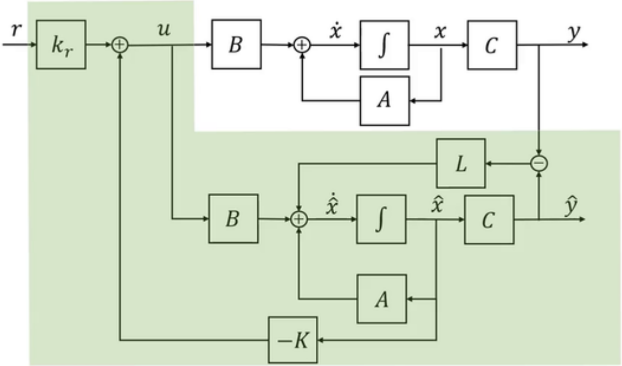

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Given the system, $\dot{x} = Ax + Bu$ , $y = Cx$, the controller, $u = -K\hat{x} + k_r r$, and the state estimator, $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$, the closed loop system can be written as:

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - CL \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} r$$

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***Rule of thumb:*** Make the estimator poles 4-5 times faster then the ”feedback” poles.

# Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

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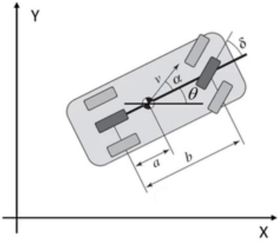


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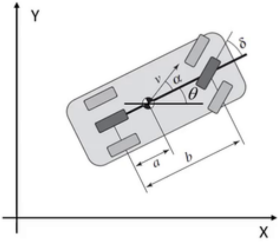


State feedback control (poles in -1 (double pole)):

$$u = -0.0278x_1 - 0.6111x_2 + 0.0278r$$

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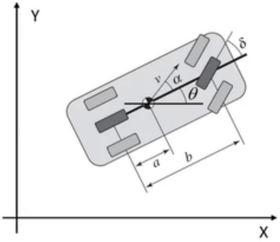


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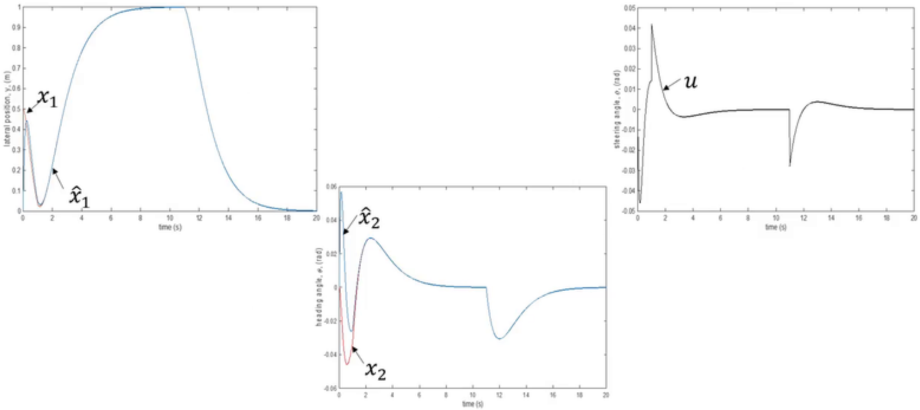
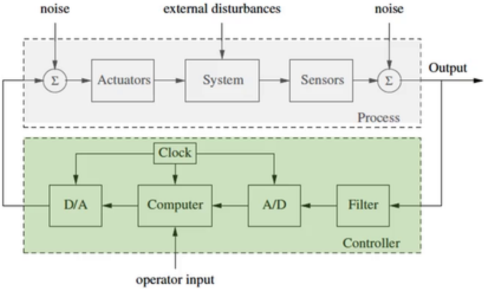
# Revisit Example - Vehicle steering (Ex 7.4)



# Implementation



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Our controller consists of the state feedback controller,

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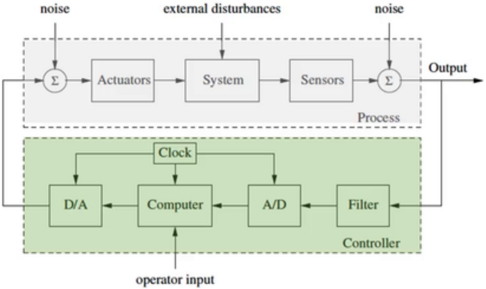

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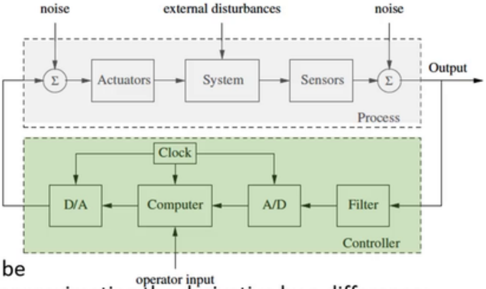


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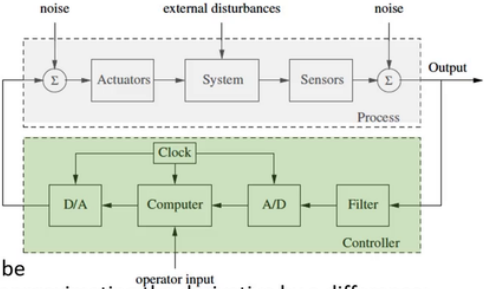


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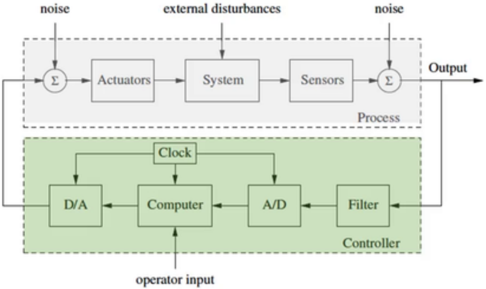
Rewriting it as a difference equation:

$$\hat{x}(t_{k+1}) = \hat{x}(t_k) + \underbrace{(t_{k+1} - t_k)}_{h\text{ - sampling time}}(A\hat{x}(t_k) + Bu(t_k) + L(y(t_k) - C\hat{x}(t_k))),$$

# Implementation




In pseudocode:

```
% Control algorithm - main loop
r = adin(ch1)                          % read reference
y = adin(ch2)                          % get process output
u = -K*xhat + kr*r                     % compute control variable
daout(ch1, u)                          % set analog output
xhat = xhat + h*(A*x+B*u+L*(y-C*x))    % update state estimate
```

# Bibliography


- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 8.