# Space feedback control

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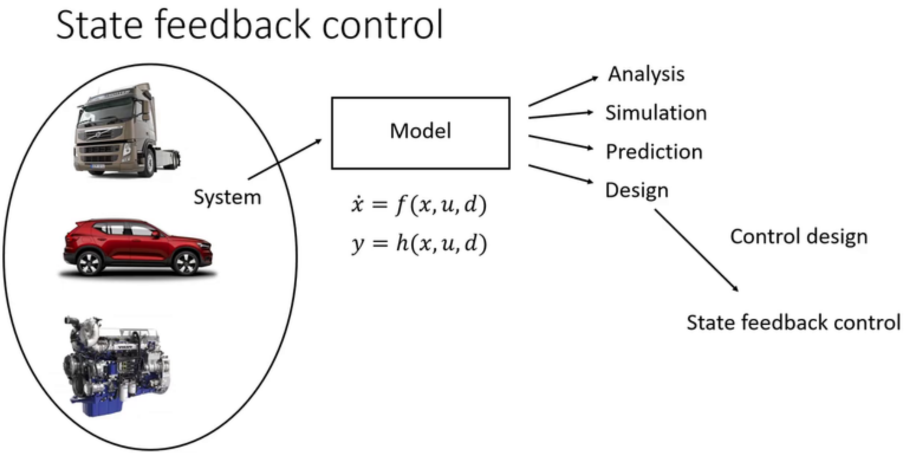
# Summary

1. State feedback control
   - Feedback gain
   - Reference tracking
   - Integral action
   - Example

2. Reachability
   - Definition
   - Revisit Example

# State feedback control



# State feedback control

Consider a linear time-invariant state-space model given by:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t) \in \mathbb{R}^n$ is the state (vector), $u(t) \in \mathbb{R}^p$ is the input or control signal and $y(t) \in \mathbb{R}^q$ is the output signal. (For SISO case, $p = 1, q = 1$)

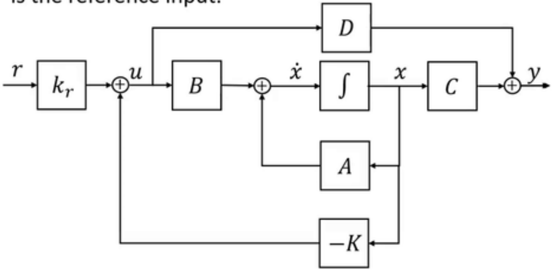
The system poles are given by the eigenvalues of the system matrix $A \in \mathbb{R}^{n \times n}$.

# State feedback control

***Idea with control design*:** Modify the eigenvalues of $A$ by using the input $u(t)$

***State feedback controller*:** $u(t) = -Kx(t) + k_r r(t)$

where $K \in \mathbb{R}^{p\times n}$ is the feedback gain, $k_r \in \mathbb{R}^{p\times r}$ is the steady-state reference gain and $r(t) \in \mathbb{R}^r$ is the reference input.



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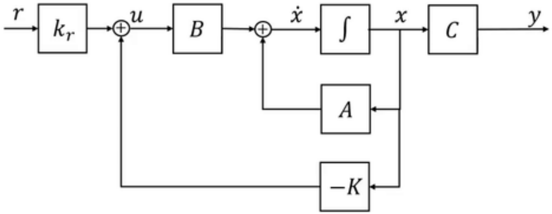

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Using the state feedback controller the closed loop dynamics becomes:

$$\dot{x}(t) = Ax(t) + B(-Kx(t) + k_r r(t))$$
$$= (A - BK)x(t) + Bk_r r(t)$$

***Control objective***: Choose $K$ such that the closed loop dynamics $A - BK$ get desired properties, i.e fulfill the specifications or stabilize the system.

SISO case: *$n$ parameters in $K$ and $n$ eigenvalues in $A$, so it might be possible!*

# Reference tracking

The steady-state reference gain, $k_r$, does not affect the stability, but it does affect the steady-state solution.

The steady-state gain is usually chosen such that:

$$y(t) \approx r(t) \;\; as\; t \to \infty$$

At steady-state the time derivative of the state variable is $\dot{x}(t) \equiv 0$, so

$$0 = (A - BK)x(t) + Bk_r r(t)$$

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$$\left. \begin{aligned} 0 &= (A - BK)x(t) + Bk_r r(t) \\ y(t) &= Cx(t) \end{aligned} \right\} \quad y = -C(A - BK)^{-1} B k_r r$$

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If $y(t) \approx r(t) \;\; as\; t \to \infty$, then $k_r$ should be chosen as

$$k_r = -(C(A - BK)^{-1}B)^{-1} \quad \text{or} \quad k_r = -1/C(A - BK)^{-1}B$$

# Integral action

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Introduce ***integral action*** to remove the steady-state error. Approach: introduce an additional state variable in our system which computes the integral of the error

$$\dot{z}(t) = y(t) - r(t)$$

The new state-space model becomes:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} Ax - Bu \\ y - r \end{bmatrix} = \begin{bmatrix} Ax - Bu \\ Cx - r \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

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Given the new state-space model, we design a controller in the usual fashion and the resulting controller becomes:

$$u(t) = -Kx(t) - K_I z(t) + k_r r(t)$$

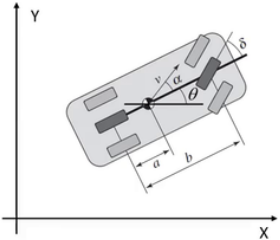
# Example 1 - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$

where $x_1$ is the lateral position Y , $x_2$ is the heading orientation $\theta$ and $u$ is the steering angle $\delta$.




The idea is to design a controller that ***stabilizes*** the dynamics and ***tracks*** a given lateral position of the vehicle.

Vehicle data: $v_0 = 12\ m/s$
$a = 2\ m$
$b = 4\ m$

***Specification:*** Desired characteristic polynomial:

$$p_{des}(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2$$

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$$\dot{x} = (A - BK)x + Bk_r r = \left( \begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} k_1 a v_0/b & k_2 a v_0/b \\ k_1 v_0/b & k_2 v_0/b \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k_r a v_0/b \\ k_r v_0/b \end{bmatrix} r$$

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The closed loop system has the characteristic polynomial

$$\det(\lambda I - A + BK) = \cdots = \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$

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Matching with desired characteristic polynomial gives:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \equiv \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$

# Example 1 - Vehicle steering (Ex 7.4)

The steady-state gain can be determined:

$$k_r = -1/C(A - BK)^{-1}B = \cdots = k_1 = \frac{b\omega_n^2}{v_0^2}$$

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The steady-state gain can be determined:

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Inserting these control design parameters into the feedback controller gives:

$$u = -k_1x_1 - k_2x_2 + k_r r = -\frac{b\omega_n^2}{v_0^2}x_1 - \left(\frac{2\zeta\omega_n b}{v_0} - \frac{ab\omega_n^2}{v_0^2}\right)x_2 + \frac{b\omega_n^2}{v_0^2}r$$

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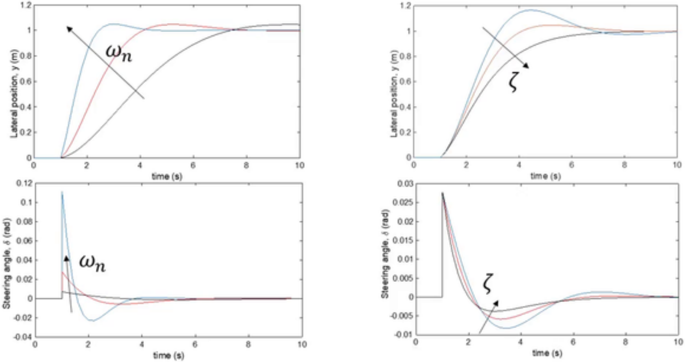
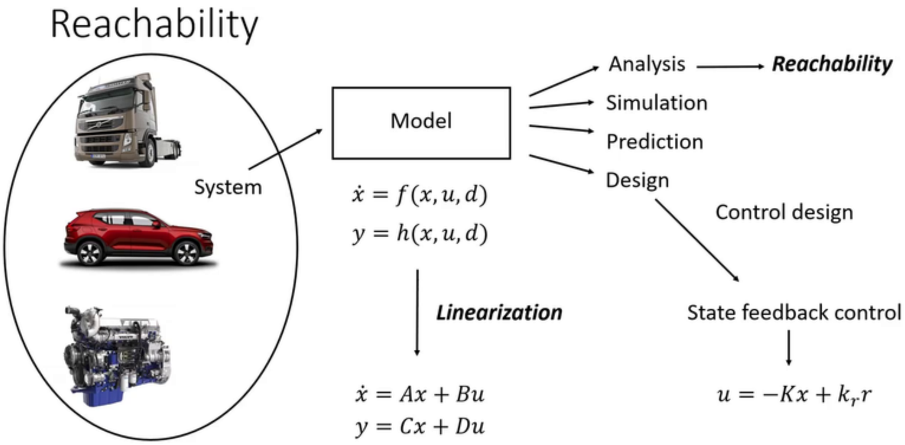
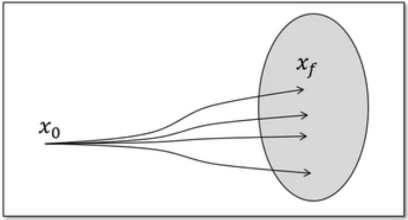
Simulations with different values of $\zeta$ and $\omega_n$:




Reachability
System
Model
$\dot{x} = f(x, u, d)$
$y = h(x, u, d)$
Analysis
Reachability
Simulation
Prediction
Design
Control design
Linearization
State feedback control
$\dot{x} = Ax + Bu$
$y = Cx + Du$
$u = -Kx + k_r r$

# Reachability

**Definition** (Reachability): A linear system is ***reachable*** if for any $x_0, x_f \in \mathbb{R}^n$ there exists a $T > 0$ and $u\colon [0,T] \to \mathbb{R}$ such that if $x(0) = x_0$ then the corresponding solution satisfies $x(T) = x_f$.



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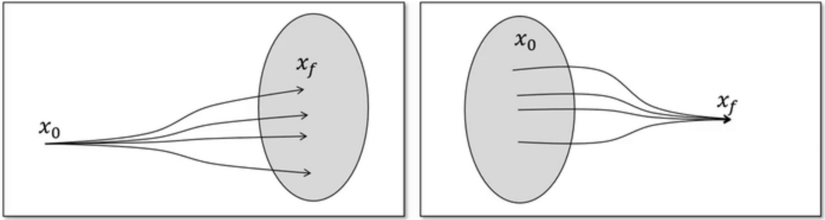
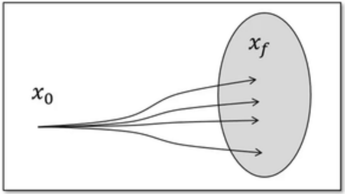



Sometimes the definition of ***controllable*** and ***controllability*** is used, and that is similar.

# Reachability




To see that an arbitrary point can be reached, we can use the convolution equation.

Assume that the system starts from zero, the state of a linear system is given by:

$$x(t) = \int_0^t e^{A(t-\tau)} B\, u(\tau) d\tau = \int_0^t e^{A\tau} B\, u(t-\tau) d\tau$$

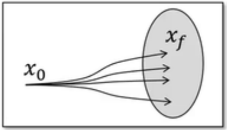
From linear theory it can be shown that

$$e^{A\tau} = I\alpha_0(\tau) + A\alpha_1(\tau) + \cdots + A^{n-1}\alpha_{n-1}(\tau)$$

where $\alpha_i(t)$ are scalar functions, so that

$$x(t) = B\int_0^t \alpha_0(\tau)\, u(t-\tau) d\tau + AB\int_0^t \alpha_1(\tau)\, u(t-\tau) d\tau + \cdots + A^{n-1}B\int_0^t \alpha_{n-1}(\tau)\, u(t-\tau) d\tau$$

# Reachability




By writing it in ”vector form”, we get:

$$x(t) = \underbrace{[B \quad AB \quad \dots \quad A^{n-1}B]}_{W_r} \begin{bmatrix} \int_0^t \alpha_0(\tau)\, u(t-\tau) d\tau \\ \int_0^t \alpha_1(\tau)\, u(t-\tau) d\tau \\ \vdots \\ \int_0^t \alpha_{n-1}(\tau)\, u(t-\tau) d\tau \end{bmatrix}$$

To reach an arbitrary point is state-space, we require that $W_r$ is nonsingular. The matrix $W_r$ is called the ***reachability matrix***.

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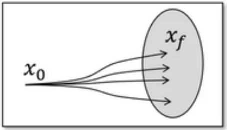


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**Theorem** (Reachability rank condition): *A linear system is reachable if and only if the reachability matrix is invertable (has full rank).*

# Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$

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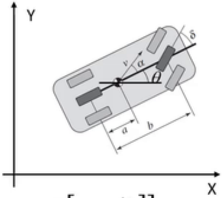


Reachability matrix:

$$W_r = [B \quad AB] = \begin{bmatrix}\begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix} & \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}\end{bmatrix}$$

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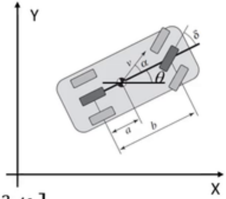


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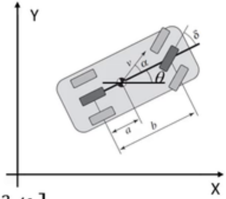


Reachability matrix:

$$W_r = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} av_0/b & v_0^2/b \\ v_0/b & 0 \end{bmatrix}$$

Compute the determinant:

$$\det(W_r) = \begin{vmatrix} av_0/b & v_0^2/b \\ v_0/b & 0 \end{vmatrix} = av_0/b \cdot 0 - v_0/b \cdot v_0^2/b = -v_0^3/b^2 \neq 0$$

*The system is reachable, as long as* $v_0 \neq 0$.

# Revisit - Example

Return to our example, with the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u$$

Reachability matrix:

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Compute the determinant:

$$\det(W_r) = \begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix} = 1 \cdot 0 - 0 \cdot 0 = 0$$

*The system is not reachable!*

# Bibliography


- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 7.