Space feedback control
Pole placement

Dr. Ing. Rodrigo Gonzalez

rodrigo.gonzalez@ingenieria.uncuyo.edu.ar

Control y Sistemas

Ingeniería Mecatrónica,
Facultad de Ingeniería,
Universidad Nacional de Cuyo

June 2020

Pole placement

So far we have learnt how a state feedback looks like and when it is possible to design a state feedback controller to stabilize a system:

$$\dot{x} = Ax + Bu$$
$$u = -Kx + k_r r$$

$$\dot{x} = (A - BK)x + Bk_r r$$

Pole placement

So far we have learnt how a state feedback looks like and when it is possible to design a state feedback controller to stabilize a system:

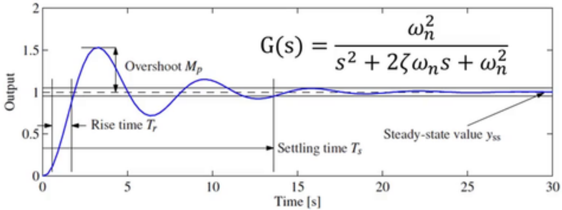
$$\dot{x} = Ax + Bu \qquad \dot{x} = (A - BK)x + Bk_r r$$

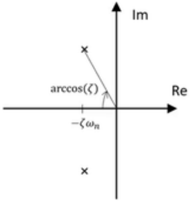
$$u = -Kx + k_r r$$

The questions that remain are: *How do we design a state feedback controller* and *Where do we place the closed loop system's poles?*

Specifications and pole placement

Property	Value	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$	$\zeta = 1$
Rise time (inverse slope)	$T_r = e^{\varphi/\tan\varphi}/\omega_0$	$1.8/\omega_0$	$2.2/\omega_0$	$2.7/\omega_0$
Overshoot	$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	16%	4%	0%
Settling time (2%)	$T_s \approx 4/\zeta\omega_0$	$8.0/\omega_0$	$5.6/\omega_0$	$4.0/\omega_0$





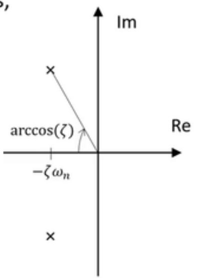
Pole placement

Where do we place the closed loop system's poles?

Idea:

- Use time domain specifications to place the dominant poles, as a second order system, $s^2 + 2\zeta\omega_n s + \omega_n^2$.

Property	Value	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$	$\zeta = 1$
Rise time (inverse slope)	$T_r = e^{\varphi/\tan\varphi}/\omega_0$	$1.8/\omega_0$	$2.2/\omega_0$	$2.7/\omega_0$
Overshoot	$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	16%	4%	0%
Settling time (2%)	$T_s \approx 4/\zeta\omega_0$	$8.0/\omega_0$	$5.6/\omega_0$	$4.0/\omega_0$

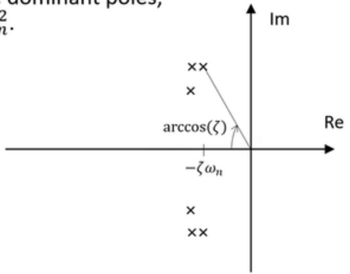


Pole placement

Where do we place the closed loop system's poles?

Idea:

- Use time domain specifications to place the dominant poles, as a second order system, $s^2 + 2\zeta\omega_n s + \omega_n^2$.
- Place the rest of the poles so they become faster than the dominant poles.



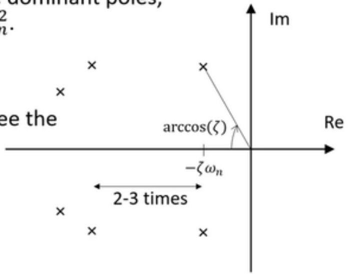
Pole placement

Where do we place the closed loop system's poles?

Idea:

- Use time domain specifications to place the dominant poles, as a second order system, $s^2 + 2\zeta\omega_n s + \omega_n^2$.
- Place the rest of the poles so they become faster than the dominant poles.

(The dominant second order poles, guarantee the time specification.)



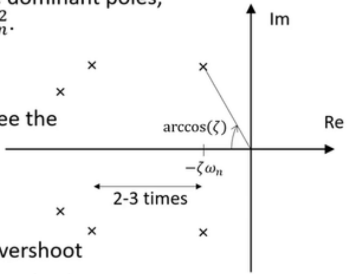
Pole placement

Where do we place the closed loop system's poles?

Idea:

- Use time domain specifications to place the dominant poles, as a second order system, $s^2 + 2\zeta\omega_n s + \omega_n^2$.
- Place the rest of the poles so they become faster than the dominant poles.

  (The dominant second order poles, guarantee the time specification.)



Usually you end up with some zeros as well:

- Zeros in the left half plane give additional overshoot
- Zeros in the right half plane give a negative undershoot

Pole placement (Ackermann's formula)

Pole placement is performed by matching the desired characteristic polynomial with the closed loop system's characteristic polynomial.

From earlier example (vehicle steering) we have seen:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \equiv \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$

$$k_1 = \frac{b\omega_n^2}{v_0^2} \qquad k_2 = \frac{2\zeta\omega_n b}{v_0} - \frac{ab\omega_n^2}{v_0^2}$$

Pole placement (Ackermann’s formula)

Pole placement is performed by matching the desired characteristic polynomial with the closed loop system’s characteristic polynomial.

From earlier example (vehicle steering) we have seen:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \equiv \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$

$$k_1 = \frac{b\omega_n^2}{v_0^2} \qquad k_2 = \frac{2\zeta\omega_n b}{v_0} - \frac{ab\omega_n^2}{v_0^2}$$

For low order systems it is ok, but for larger systems this is boring work.

Ackermann’s formula offers us a method to do this in one computational step.

Pole placement (Ackermann's formula)

Consider a system $\dot{x} = Ax + Bu$ with the characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + \cdots a_{n-1} s + a_n.$$

Pole placement (Ackermann's formula)

Consider a system $\dot{x} = Ax + Bu$ with the characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + \cdots a_{n-1} s + a_n.$$

If the system is reachable, then there exist a control law, $u = -Kx$, that gives a closed loop system with the charateristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots p_{n-1} s + p_n.$$

Pole placement (Ackermann's formula)

The feedback gain is given by

$$K = [p_1 - a_1 \quad p_2 - a_2 \quad \dots \quad p_n - a_n]\tilde{W}_r W_r^{-1}$$

where W_r is the reachability matrix

$$W_r = [B \quad AB \quad \dots \quad A^{n-1}B]$$

and

$$\tilde{W}_r = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}^{-1}$$

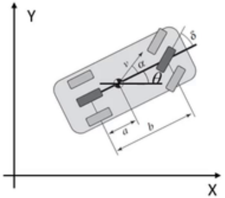
This is called ***Ackermann's formula***.

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$

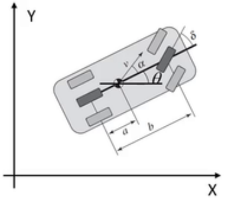


Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$



Determine the characteristic polynomial for the system:

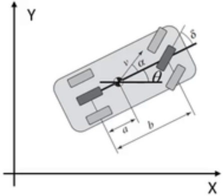
$$\det(sI - A) = \begin{vmatrix}s & -v_0\\ 0 & s\end{vmatrix} = s^2 = s^2 + 0s + 0$$

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$



Determine the characteristic polynomial for the system:

$$\det(sI - A) = \begin{vmatrix}s & -v_0\\ 0 & s\end{vmatrix} = s^2 = s^2 + 0s + 0$$

Desired characteristic polynomial for the closed loop system:

$$p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Revisit Example - Vehicle steering (Ex 7.4)

The feedback gain is given by

$$K = [p_1 - a_1 \quad p_2 - a_2]\widetilde{W}_r W_r^{-1} = [2\zeta\omega_n \quad \omega_n^2]\widetilde{W}_r W_r^{-1}$$

Revisit Example - Vehicle steering (Ex 7.4)

The feedback gain is given by

$$K = [p_1 - a_1 \quad p_2 - a_2]\tilde{W}_r W_r^{-1} = [2\zeta\omega_n \quad \omega_n^2]\tilde{W}_r W_r^{-1}$$

where W_r is the reachability matrix

$$W_r = [B \quad AB] = \begin{bmatrix} av_0/b & v_0^2/b \\ v_0/b & 0 \end{bmatrix}$$

and

$$\tilde{W}_r = \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Revisit Example - Vehicle steering (Ex 7.4)

The feedback gain is given by

$$K = [p_1 - a_1 \quad p_2 - a_2]\tilde{W}_r W_r^{-1} = [2\zeta\omega_n \quad \omega_n^2]\tilde{W}_r W_r^{-1}$$

where W_r is the reachability matrix

$$W_r = [B \quad AB] = \begin{bmatrix} av_0/b & v_0^2/b \\ v_0/b & 0 \end{bmatrix}$$

and

$$\tilde{W}_r = \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The feedback gain, can then be computed as

$$K = [2\zeta\omega_n \quad \omega_n^2]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} av_0/b & v_0^2/b \\ v_0/b & 0 \end{bmatrix}^{-1}$$

Revisit Example - Vehicle steering (Ex 7.4)

The feedback gain is given by

$$K = [p_1 - a_1 \quad p_2 - a_2]\widetilde{W}_r W_r^{-1} = [2\zeta\omega_n \quad \omega_n^2]\widetilde{W}_r W_r^{-1}$$

where W_r is the reachability matrix

$$W_r = [B \quad AB] = \begin{bmatrix} av_0/b & v_0^2/b \\ v_0/b & 0 \end{bmatrix}$$

and

$$\widetilde{W}_r = \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The feedback gain, can then be computed as

$$K = [2\zeta\omega_n \quad \omega_n^2]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} av_0/b & v_0^2/b \\ v_0/b & 0 \end{bmatrix}^{-1} = [2\zeta\omega_n \quad \omega_n^2]\begin{bmatrix} 0 & b/v_0 \\ b/v_0^2 & -ab/v_0^2 \end{bmatrix}$$

Revisit Example - Vehicle steering (Ex 7.4)

The feedback gain is given by

$$K = [p_1 - a_1 \quad p_2 - a_2]\widetilde{W}_r W_r^{-1} = [2\zeta\omega_n \quad \omega_n^2]\widetilde{W}_r W_r^{-1}$$

where W_r is the reachability matrix

$$W_r = [B \quad AB] = \begin{bmatrix} av_0/b & v_0^2/b \\ v_0/b & 0 \end{bmatrix}$$

and

$$\widetilde{W}_r = \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The feedback gain, can then be computed as

$$K = [2\zeta\omega_n \quad \omega_n^2]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} av_0/b & v_0^2/b \\ v_0/b & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \underbrace{\frac{b\omega_n^2}{v_0^2}}_{k_1} & \underbrace{\frac{2\zeta\omega_n b}{v_0} - \frac{ab\omega_n^2}{v_0^2}}_{k_2} \end{bmatrix}$$

- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 7.