Space feedback control

Linear Quadratic Regulator (LQR control)

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June 2020

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where S is a positive definite, symmetric matrix given by

$$A^T S + SA - SBQ_u^{-1} B^T S + Q_x = 0$$

This equation is called the ***algebraic Riccati equation***.

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The tuning of the LQR is to choose the weighting matrices Q_x and Q_u. To guarantee that a solution exists, the system must be ***reachable*** and that $\boldsymbol{Q_x} \succcurlyeq \mathbf{0}$ and $\boldsymbol{Q_u} \succ \mathbf{0}$.

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2. Output weighting. Let $z = C_z x$ be the output you want to keep small.

   Choose $Q_x = C_z^T C_z$, and $Q_u = \rho I$. $\quad \Longrightarrow \quad$ trade-off $\Longrightarrow \|z\|^2 \;\; vs \;\; \rho \|u\|^2$

Linear Quadratic Regulator

3. Diagnonal weighting.

$$Q_x = \begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{bmatrix} \quad Q_u = \begin{bmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_p \end{bmatrix}$$

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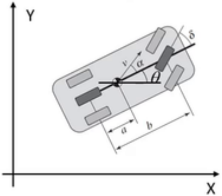
4. Trial and error

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$



Vehicle data: $v_0 = 12\, m/s$

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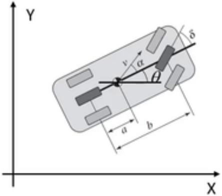

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Place the poles so that the closed loop system optimizes the cost function:

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where

$$Q_x = \begin{bmatrix}q_1 & 0\\ 0 & q_2\end{bmatrix} \qquad Q_u = \rho$$

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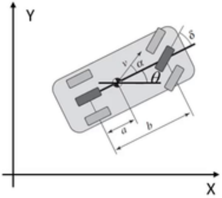
For the case when

$$Q_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q_u = 10$$



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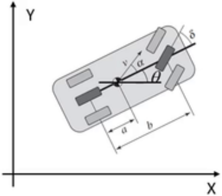

In MATLAB: The LQR problem can be solved using the `lqr` command

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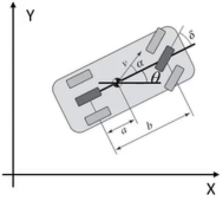
For the case when

$$Q_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q_u = 10$$

The solution to the algebraic Ricatti equation is

$$S = \begin{bmatrix} 0.292 & 0.470 \\ 0.470 & 2.754 \end{bmatrix}$$



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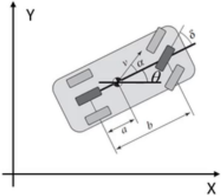

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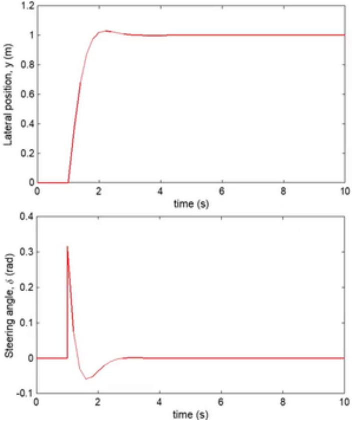

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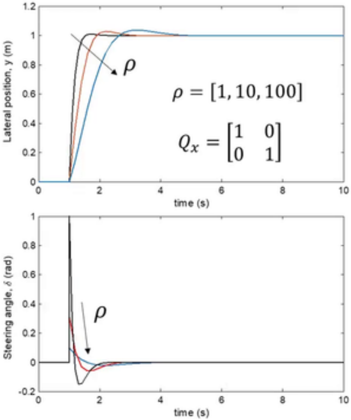

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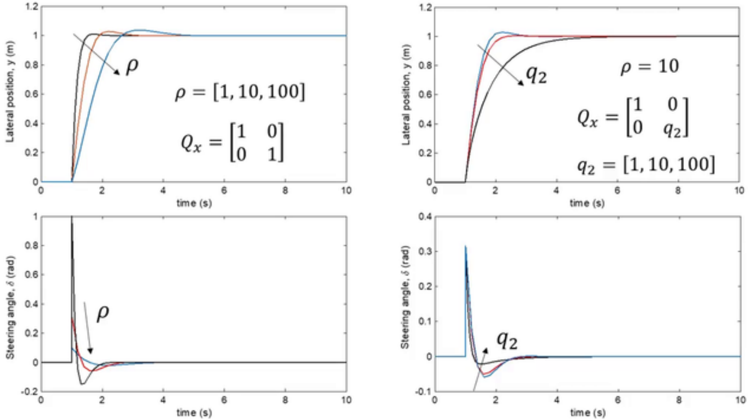

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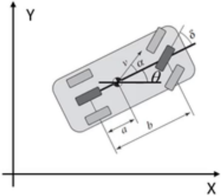

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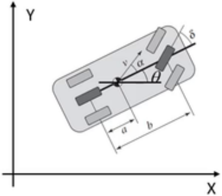
Compared to the pole placement design, this corresponds to $\zeta = 0.77$ and $\omega_n = 3.44$.



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Bibliography

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