Observability

Dr. Ing. Rodrigo Gonzalez

rodrigo.gonzalez@ingenieria.uncuyo.edu.ar

Control y Sistemas

Ingeniería Mecatrónica,
Facultad de Ingeniería,
Universidad Nacional de Cuyo

June 2020

Observability

Definition (Observability): A linear system is ***observable*** if for every $T > 0$ it is possible to determine the state of the system $x(T)$ through the measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$.

Recall the solution to the differential equation:

$$y(t) = Ce^{At}x(0)$$

Since we know $u(t)$, we only needs to consider zero input case.

So, if $x(0)$ can be determined, then we can reconstruct $x(t)$ exactly.

Observability

Definition: A ***state*** $x^* \neq 0$ is said to be ***unobservable*** if the zero-input solution $y(t) = Ce^{At}x(0)$, with $x(0) = x^*$, is zero for all $t \geq 0$.

Observability

Definition: A ***state*** $x^* \neq 0$ is said to be ***unobservable*** if the zero-input solution $y(t) = Ce^{At}x(0)$, with $x(0) = x^*$, is zero for all $t \geq 0$.

So, if we can find a state x^* for which $Ce^{At}x(0) = 0$ for all $t \geq 0$, then the system is unobservable. For this to hold, all derivatives must be zero at $t = 0$.

Observability

Definition: A ***state*** $x^* \neq 0$ is said to be ***unobservable*** if the zero-input solution $y(t) = Ce^{At}x(0)$, with $x(0) = x^*$, is zero for all $t \geq 0$.

So, if we can find a state x^* for which $Ce^{At}x(0) = 0$ for all $t \geq 0$, then the system is unobservable. For this to hold, all derivatives must be zero at $t = 0$.

$$
\begin{aligned}
Ce^{At}x^*\Big|_{t=0} &= 0 \quad \Longrightarrow \quad Cx^*=0 \\
\frac{d}{dt}Ce^{At}x^*\Big|_{t=0} &= 0 \quad \Longrightarrow \quad CAe^{At}x^*\big|_{t=0} = CAx^*=0 \\
\frac{d^2}{dt^2}Ce^{At}x^*\Big|_{t=0} &= 0 \quad \Longrightarrow \quad CA^2e^{At}x^*\big|_{t=0} = CA^2x^*=0 \\
&\;\;\vdots \\
\frac{d^{n-1}}{dt^{n-1}}Ce^{At}x^*\Big|_{t=0} &= 0 \quad \Longrightarrow \quad CA^{n-1}e^{At}x^*\big|_{t=0} = CA^{n-1}x^*=0
\end{aligned}
$$

Observability

Definition: A ***state*** $x^* \neq 0$ is said to be ***unobservable*** if the zero-input solution $y(t) = Ce^{At}x(0)$, with $x(0) = x^*$, is zero for all $t \geq 0$.

So, if we can find a state x^* for which $Ce^{At}x(0) = 0$ for all $t \geq 0$, then the system is unobservable. For this to hold, all derivatives must be zero at $t = 0$.

$$\left.\begin{array}{lcl}
Ce^{At}x^*\Big|_{t=0} = 0 & \Longrightarrow & Cx^*=0 \\
\frac{d}{dt}Ce^{At}x^*\Big|_{t=0} = 0 & \Longrightarrow & CAe^{At}x^*\big|_{t=0} = CAx^*=0 \\
\frac{d^2}{dt^2}Ce^{At}x^*\Big|_{t=0} = 0 & \Longrightarrow & CA^2e^{At}x^*\big|_{t=0} = CA^2x^*=0 \\
 & \vdots & \\
\frac{d^{n-1}}{dt^{n-1}}Ce^{At}x^*\Big|_{t=0} = 0 & \Longrightarrow & CA^{n-1}e^{At}x^*\big|_{t=0} = CA^{n-1}x^*=0
\end{array}\right\} \quad \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{W_o} x^*=0$$

Observability

Theorem (Observability rank condition): *A linear system is observable if and only if the observability matrix reachability W_o*

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full row rank.

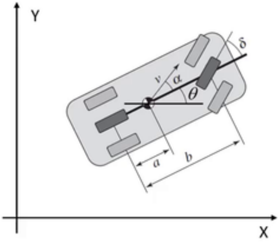
Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u$$

where x_1 is the lateral position Y , x_2 is the heading orientation θ and u is the steering angle δ.



Vehicle data: $v_0 = 12\ m/s$

$a = 2\ m$

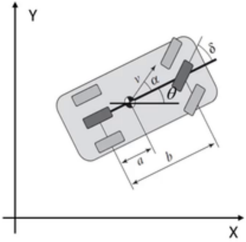
$b = 4\ m$

Example - Vehicle steering (Ex 7.4)

Return to our vehicle steering example, with lateral position as output signal:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u$$



Observability matrix:

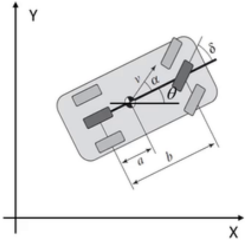
$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

Example - Vehicle steering (Ex 7.4)

Return to our vehicle steering example, with lateral position as output signal:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 12\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}6\\ 3\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$



Observability matrix:

$$W_o = \begin{bmatrix}C\\ CA\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 12\end{bmatrix}$$

Example - Vehicle steering (Ex 7.4)

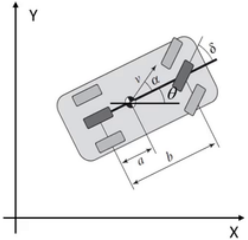
Return to our vehicle steering example, with lateral position as output signal:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u$$

Observability matrix:

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 12 \end{bmatrix}$$



Note: A square matrix M (*n*×*n*) has full rank *n* iff the det(M)≠0

Example - Vehicle steering (Ex 7.4)

Return to our vehicle steering example, with lateral position as output signal:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 12\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}6\\ 3\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$

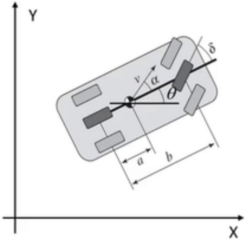
Observability matrix:

$$W_o = \begin{bmatrix}C\\ CA\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 12\end{bmatrix}$$

Compute the determinant:

$$\det(W_o) = \begin{vmatrix}1 & 0\\ 0 & 12\end{vmatrix} = 1\cdot 12 - 0\cdot 0 = 12$$

The system is observable!



Note: A square matrix M (n×n) has full rank n iff the det(M)≠0

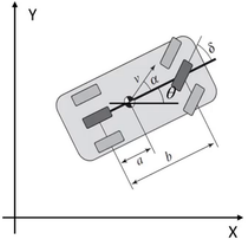
Observability

Example - Vehicle steering (Ex 7.4)

Return to our vehicle steering example, with heading angle as output signal:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u$$



Observability matrix:

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Compute the determinant:

$$\det(W_o) = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 0 = 0$$

The system is not observable!

Observability

Definition: A ***state*** $x^* \neq 0$ is said to be ***unobservable*** if the zero-input solution $y(t) = Ce^{At}x(0)$, with $x(0) = x^*$, is zero for all $t \geq 0$.

So, look at our vehicle steering example with heading angle as output:

$$\begin{bmatrix} C \\ CA \end{bmatrix} x^* = 0 \quad \Longrightarrow \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x^* = 0 \quad \Longrightarrow \quad x^* = \begin{bmatrix} x_1(0) \neq 0 \\ 0 \end{bmatrix}$$

Bibliography

- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 8.