14 INTRODUCTION TO TIME-DEPENDENT MATERIAL BEHAVIOR

It is an everyday experience that even if a material is loaded by a constant load, its deformation may increase with time; a book-shelf loaded by too heavy books may increase its deflection as years goes by. As a consequence, one speaks of *time-dependent material behavior* also termed *creep behavior*. The terminology in the literature is not unique and, traditionally, when the time-dependent strains are related linearly to the stresses, one speaks of *viscoelasticity* whereas the notations of *creep* and *viscoplasticity* are often used when the time-dependent strains depend nonlinearly on the stresses.

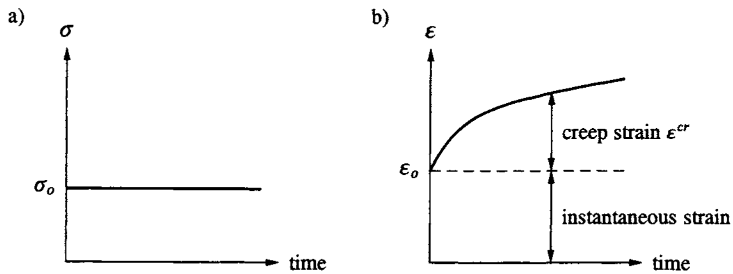

Figure 14.1: Creep test; a) stress history, b) strain history.

In this chapter, we will deal with viscoelasticity, whereas creep and viscoplasticity will be addressed in the next chapter. However, we will start with a general discussion of various experimental findings relating to time-dependent behavior.

There are three standard tests used to identify the time-dependent response of a material: the *creep test*, the *relaxation test* and the *constant strain-rate test*. In the creep test, the stress σ_0 is applied instantaneously and then kept constant, cf. Fig. 14.1a), and as a result the strain history may vary as shown in

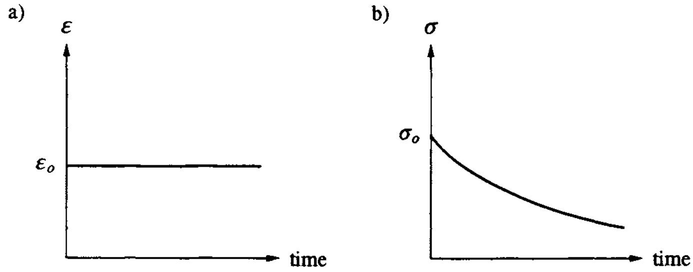

Figure 14.2: Relaxation test; a) strain history, b) stress history.

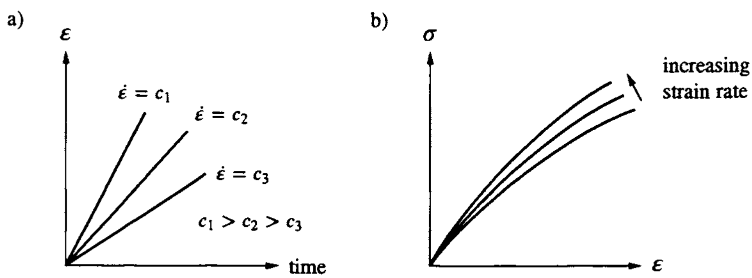

Figure 14.3: Constant strain-rate test; a) strain history for three tests, b) corresponding stress-strain responses.

Fig. 14.1b). In Fig. 14.1b), the strain ε_0 is the instantaneous strain that may be elastic (then $\varepsilon_0 = \sigma_0/E$) or elasto-plastic (then $\varepsilon_0 = \sigma_0/E + \varepsilon^p$) and with time the *creep strain* ε^{cr} develops.

Historically, the first quantitative statements about creep were made by the French engineer Vicat (1834), who observed that bridges suspended by hardened iron cables deflected significantly beyond their elastic deflections. For such cables, Vicat performed creep tests similar to the one shown in Fig. 14.1.

In the *relaxation test*, the total strain is applied instantaneously and then kept constant at the value ε_0, cf. Fig. 14.2a), and as a result the stress history may vary as shown in Fig. 14.2b). In Fig. 14.2b), the stress σ_0 is the instantaneous stress that may be a result of elastic (then $\sigma_0 = E\varepsilon_0$) or elasto-plastic response (then $\sigma_0 = E(\varepsilon_0 - \varepsilon^p)$) when enforcing the instantaneous strain ε_0. As time goes by, Fig. 14.2b) shows that the stress gradually decreases - it relaxes.

In the *constant strain-rate test*, the total strain rate $\dot{\varepsilon} = d\varepsilon/dt$ =constant is enforced on the material and the stress response is then measured so that the stress-strain relation can be established. In Fig. 14.3, the results of three such tests are shown and it appears that the larger the total strain-rate, the stiffer the material behaves. This is a characteristic property for materials that exhibit

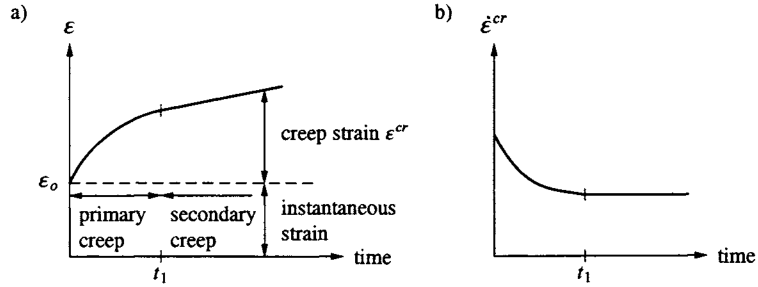

Figure 14.4: Creep test for a 'small' stress; a) strain history, b) creep strain rate $\dot{\varepsilon}^{cr}$.

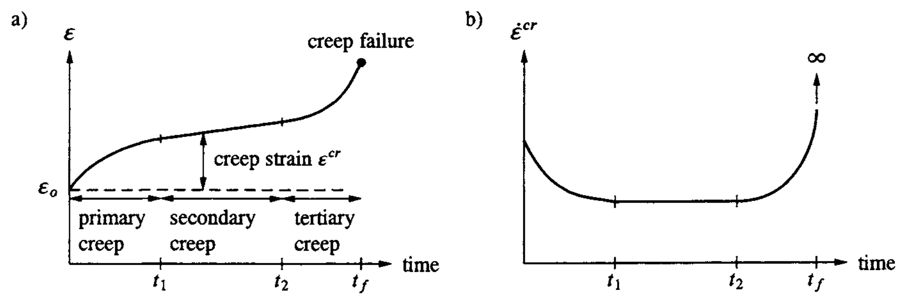

Figure 14.5: Creep test for a 'large' stress; a) strain history, b) creep strain rate $\dot{\varepsilon}^{cr}$.

creep deformation and it is concluded that for such materials, the response is *rate-dependent*. This is in striking contrast to elasto-plasticity where the response is independent of the rate applied.

If the stress applied in a creep test is not too large, the response shown in Fig. 14.4a) is obtained. Up until the time t_1, we have *primary creep* - also called *transient creep* - and after time t_1 we have *secondary creep* - also called *stationary creep*. In Fig. 14.4b), the creep strain rate $\dot{\varepsilon}^{cr} = d\varepsilon^{cr}/dt$ is shown and it appears that during primary or transient creep, $\dot{\varepsilon}^{cr}$ is decreasing whereas during secondary or stationary creep, $\dot{\varepsilon}^{cr}$ is constant. For some materials, the primary creep region is small and may be ignored; this is often the case for metals and steels exposed to high temperatures and constant load.

If the stress applied in a creep test is sufficiently large, the response shown in Fig. 14.5a) is obtained. Now we also obtain *tertiary creep* after time t_2 and in this region the creep strain rate $\dot{\varepsilon}^{cr}$ increases, cf. Fig. 14.5b), and at time t_f the material fails - *creep failure* has occurred. Modeling of the phenomenon of creep failure is very complex and we will here only be concerned with modeling of primary and secondary creep.

Let us next discuss linear and nonlinear creep response. In Fig. 14.6, the results of two creep tests are shown; one with the constant stress σ_0 and another

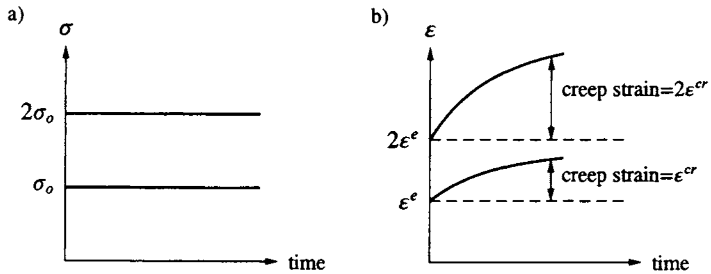

Figure 14.6: Creep test where linearity holds; if the stress is doubled the creep strain is also doubled.

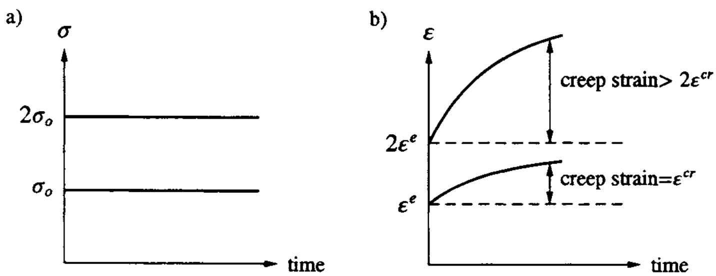

Figure 14.7: Creep test when nonlinearity holds; if the stress is doubled the creep strain is more than doubled.

test with twice the stress, i.e. $2\sigma_0$. For simplicity, the instantaneous response is assumed to be linear elastic, i.e. if it is ε^e in the first test it is $2\varepsilon^e$ in the second test. For linearity to hold, then if ε^{cr} is the creep strain in the first test at some time, the creep strain in the second test should be $2\varepsilon^{cr}$ at the same time. This linearity is characteristic for *viscoelasticity*, which will be discussed in the next section. Viscoelastic response is typical for polymers and concrete loaded not too close to their ultimate strength.

To illustrate nonlinear creep response, Fig. 14.7 is considered. Again two creep tests are performed, one with the stress σ_0 and the other with the stress $2\sigma_0$. For simplicity, the instantaneous response is again assumed to be linear elastic, i.e. the instantaneous strain is ε^e in the first test and $2\varepsilon^e$ in the second test. Now however, the creep strain in the second test is not twice the creep strain in the first test; in practice it is larger. This nonlinear creep response is typical for creep of metals and steel and we shall discuss various means to model such creep response in the next chapter.

To illustrate the phenomenon of *recovery*, the creep test in Fig. 14.8 is considered. As shown in Fig. 14.8a), the stress σ_0 is removed at time t_1 and after that

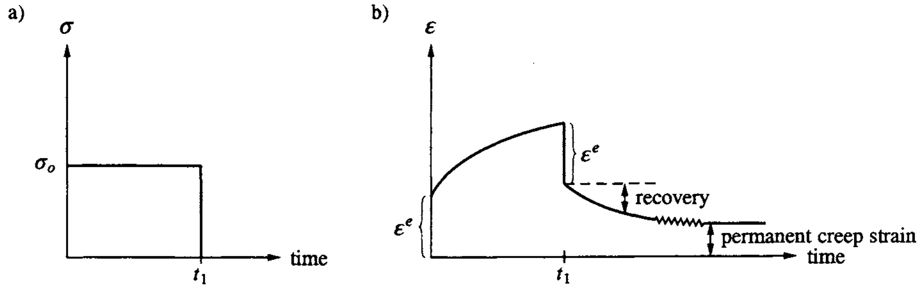

Figure 14.8: Phenomenon of recovery.

the material is completely unloaded. The corresponding strain development is shown in Fig. 14.8b) where - for simplicity - the instantaneous strain is assumed to be elastic, i.e. equal to ε^e. Figure 14.8b) shows that when the specimen is unloaded at time t_1, it responds elastically and thereby the strain decreases with the amount ε^e at time t_1; however, even though the material is unstressed after time t_1, the strain continues to decrease, i.e. the strain - or some part of it - is recovered. This phenomenon is characteristic for viscoelastic materials like polymers and concrete. The part of the total strain that is not recovered even after infinitely long time is called *permanent creep strain.*

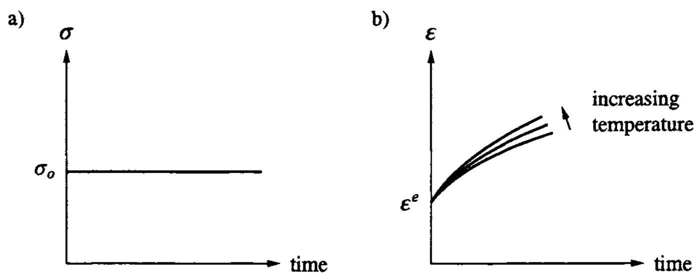

Figure 14.9: Influence of temperature.

In practice, it turns out that development of creep strains is very sensitive to temperature; the higher the temperature, the larger the creep strain. This is illustrated in Fig. 14.9 where - for simplicity - the instantaneous response is assumed to be linear elastic and where the E–modulus is assumed to be temperature-independent. We shall later return to a more detailed discussion of the influence of temperature on various materials.

14.1 Viscoelasticity

It is recalled that viscoelasticity means that if the stress in a creep test is doubled then the total strain is also doubled, cf. Fig. 14.6; this linearity is also assumed to hold for general load histories. The term viscoelasticity is used since the behavior is something between that of a viscous fluid and an elastic solid. Especially in older literature and in material science literature, viscoelastic models are also referred to as *rheological models* and strictly speaking *rheology* means the science of viscous fluids.

If the loading is such that the response - apart from being time-dependent - is elastic, the linearity property that is characteristic for viscoelasticity is closely fulfilled for polymers, cf. for instance Finnie and Heller (1959), Bartenev and Zuev (1968), Williams (1980) and Mills (1986), and for concrete, cf. for instance Finnie and Heller (1959), Neville (1963), Hannant (1969) and Browne and Blundell (1972).

It turns out that there are two routes that can be followed in order to model viscoelasticity; one is the *differential approach* and the other is the *hereditary approach*. We will now provide a brief introduction to these formulations and the reader is referred to, for instance, Flügge (1967), Hunter (1983), Malvern (1969), Pipkin (1972), Findley *et al.* (1976), Rabotnov (1980) and Williams (1980) for further information.

As emphasized above, we take the linearity principle as a basis for viscoelasticity and this leads to linear viscoelasticity that will be discussed below. Linear viscoelasticity has been successfully applied to concrete, most polymers, wood and paper. However, it will turn out that this formulation even makes for the possibility of modeling nonlinear viscoelasticity.

Let us first introduce the *creep compliance* $J(t)$ according to the following definition

$$\boxed{\begin{array}{l}\textit{The creep compliance } J(t)\\ = \textit{strain developed in a creep test}\\ \quad \textit{when loaded by a unit stress}\end{array}} \tag{14.1}$$

Since the linearity principle holds, the strain development in a creep test with the constant stress σ_0 is then $\varepsilon(t) = J(t)\sigma_0$. Comparing with Fig. 14.1, the creep compliance function $J(t)$ therefore gives information on how the strain develops with time in a creep test.

In a similar manner, the *relaxation modulus* $G(t)$ is defined by

$$\boxed{\begin{array}{l}\textit{The relaxation modulus } G(t)\\ = \textit{stress developed in a relaxation test}\\ \quad \textit{when loaded by a unit strain}\end{array}} \tag{14.2}$$

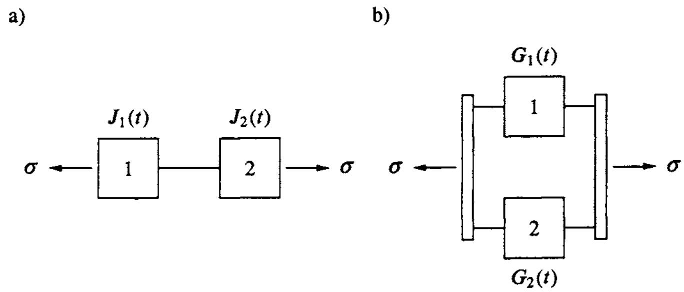

Figure 14.10: Two viscoelastic models in a) series and b) parallel.

where the relaxation modulus $G(t)$ should not be confused with the shear modulus G related to linear elasticity. Since the linearity principle holds, the stress development in a relaxation test with the constant strain ε_0 is then given by $\sigma(t) = G(t)\varepsilon_0$. Comparing with Fig. 14.2, the relaxation modulus function $G(t)$ therefore gives information on how the stress develops with time in a relaxation test. Certainly, one may expect that, in some fashion, the creep compliance $J(t)$ and the relaxation modulus $G(t)$ are related and we will establish this relation in the section dealing with the hereditary approach.

Suppose that two viscoelastic models with creep compliances $J_1(t)$ and $J_2(t)$ are placed in series, cf. Fig. 14.10a). With evident notation, we have

$$\sigma = \sigma_1 = \sigma_2\,; \qquad \varepsilon = \varepsilon_1 + \varepsilon_2$$

In a creep test with the stress σ_0, it follows that $\varepsilon = J_1(t)\sigma_1 + J_2(t)\sigma_2 = (J_1(t) + J_2(t))\sigma_0$. It is concluded that

For two models placed in series

$$J(t) = J_1(t) + J_2(t) \tag{14.3}$$

holds

Indeed, in a relaxation test it is also possible to establish how $G(t)$ is related to $G_1(t)$ and $G_2(t)$, but in the following we will not make use of this slightly more complex relation.

Consider next that two viscoelastic models with relaxation moduli $G_1(t)$ and $G_2(t)$ are placed in parallel, cf. Fig. 14.10b). It follows that

$$\sigma = \sigma_1 + \sigma_2\,; \qquad \varepsilon = \varepsilon_1 = \varepsilon_2$$

In a relaxation test with the strain ε_0, we then obtain $\sigma = G_1(t)\varepsilon_1 + G_2(t)\varepsilon_2 =$

$(G_1(t) + G_2(t))\varepsilon_0$. Consequently

For two models placed in parallel
$G(t) = G_1(t) + G_2(t)$
holds

(14.4)

In a creep test, it is also possible to establish how $J(t)$ is related to $J_1(t)$ and $J_2(t)$, but in the following we will not make use of this slightly more complex relation.

14.2 Differential equation approach

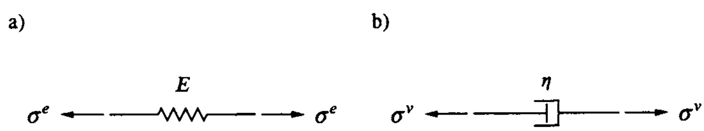

Figure 14.11: a) Linear spring; b) dashpot.

In the differential approach, viscoelastic models are constructed by various combinations of linear springs and dashpots. We have already touched upon this approach in Section 6.4 and the constitutive equations that control the spring and the dashpot shown in Fig. 14.11 are

$$\sigma^e = E\varepsilon^e\,; \qquad \sigma^v = \eta\dot{\varepsilon}^v \tag{14.5}$$

where superscript e and v refer to elastic and viscous behavior, respectively; the *viscosity coefficient* η has the dimension [Pa·s].

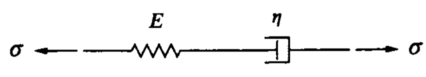

Figure 14.12: Maxwell model.

The *Maxwell model* - established by Maxwell (1868) - is shown in Fig. 14.12 and it consists of a spring and dashpot in series. It appears that

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^v\,; \qquad \sigma = \sigma^e = \sigma^v \tag{14.6}$$

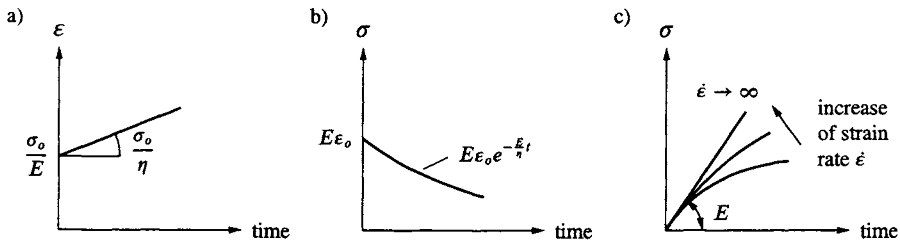

Figure 14.13: Response of Maxwell model; a) creep test, b) relaxation test and c) constant strain-rate test.

Insertion of (14.5) in (14.6a) and use of (14.6b) give the following constitutive equation for the Maxwell model

$$\boxed{\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \qquad \textit{Maxwell model}} \tag{14.7}$$

To determine the response during a creep test, (14.7) is multiplied by dt and integrated. This gives $\varepsilon = \frac{\sigma}{E} + \frac{1}{\eta}\int_0^t \sigma(\tau)d\tau = (\frac{1}{E} + \frac{t}{\eta})\sigma_0$ and a comparison with (14.1) shows that the creep compliance is $J(t) = \frac{1}{E} + \frac{t}{\eta}$. The response to a creep test is shown in Fig. 14.13a) and it appears that the Maxwell model exhibits secondary creep. For a relaxation test, we have $\varepsilon = \varepsilon_0$ = constant and (14.7) then reduces to $\dot{\sigma} + \frac{E}{\eta}\sigma = 0$ with the solution $\sigma = Ce^{-\frac{E}{\eta}t}$ where the arbitrary constant C is determined from the condition that $t = 0$ gives $\sigma = E\varepsilon_0$, i.e. $\sigma = E\varepsilon_0 e^{-\frac{E}{\eta}t}$ as illustrated in Fig. 14.13b). Moreover, a comparison with (14.2) shows that the relaxation modulus is $G(t) = Ee^{-\frac{E}{\eta}t}$. Therefore

$$\boxed{J(t) = \frac{1}{E} + \frac{t}{\eta}; \qquad G(t) = Ee^{-\frac{E}{\eta}t} \qquad \textit{Maxwell model}} \tag{14.8}$$

For a constant strain-rate test, we have $\varepsilon = \dot{\varepsilon}t$ where the constant $\dot{\varepsilon}$ is the strain-rate. Insertion into (14.7) and integration gives $\sigma = \dot{\varepsilon}\eta + Ce^{-\frac{E}{\eta}t}$ where the arbitrary constant C is determined from the condition that $t = 0$ gives $\sigma = 0$, i.e. $\sigma = \dot{\varepsilon}\eta(1 - e^{-\frac{E}{\eta}t})$. Since $t = \varepsilon/\dot{\varepsilon}$, the stress-strain relation becomes $\sigma = \dot{\varepsilon}\eta(1 - e^{-\frac{E}{\eta}\frac{\varepsilon}{\dot{\varepsilon}}})$, which implies $\frac{d\sigma}{d\varepsilon} = Ee^{-\frac{E}{\eta}\frac{\varepsilon}{\dot{\varepsilon}}}$ and this means that the initial slope is always given as $\frac{d\sigma}{d\varepsilon} = E$ as shown in Fig. 14.13c). It appears that the stress-strain response depends on the strain-rate and that, for an infinitely large strain-rate, the response approaches linear elasticity. Considering Fig. 14.12, this is certainly not surprising since the dashpot responds as a rigid member for a sudden application of the load.

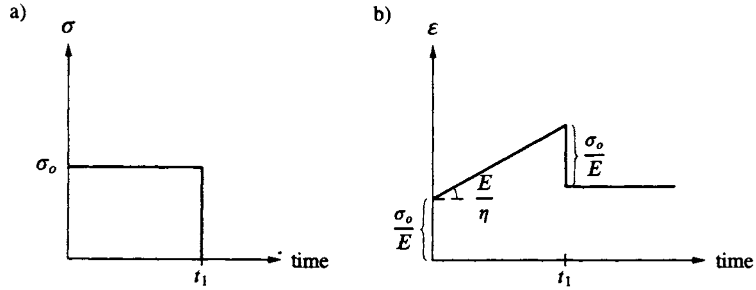

Figure 14.14: Response of Maxwell model; no recovery effect.

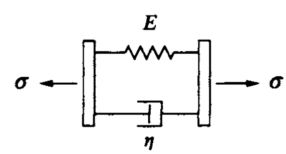

Figure 14.15: Kelvin model.

If the stress in a creep test is removed at time t_1, the Maxwell model reacts elastically during the unloading process and since $\sigma = \dot{\sigma} = 0$ when $t > t_1$, (14.7) then gives that $\dot{\varepsilon} = 0$ when $t > t_1$. We then obtain the response shown in Fig. 14.14; the Maxwell model shows no recovery, cf. Fig. 14.8.

The *Kelvin model* - established by Kelvin (1875) and also by Voigt (1892) and therefore also called the *Voigt model* - is shown in Fig. 14.15 and it consists of a spring and a dashpot in parallel. It follows that

$$\varepsilon = \varepsilon^e = \varepsilon^v ; \qquad \sigma = \sigma^e + \sigma^v \tag{14.9}$$

Insertion of (14.5) into (14.9b) and use of (14.9a) give the following constitutive equation

$$\boxed{\sigma = E\varepsilon + \eta\dot{\varepsilon} \qquad \textit{Kelvin model}} \tag{14.10}$$

To determine the response in a creep test with the constant stress σ_0, (14.10) is integrated to give $\varepsilon = \frac{\sigma_0}{E} + Ce^{-\frac{E}{\eta}t}$ where C is an arbitrary constant. Since the dashpot reacts as a rigid member when a load is suddenly applied, we have the condition $\varepsilon = 0$ when $t = 0$. This condition determines C and we then obtain $\varepsilon = \frac{\sigma_0}{E}(1 - e^{-\frac{E}{\eta}t})$ as shown in Fig. 14.16a). From (14.1) it is concluded that

$$\boxed{J(t) = \frac{1}{E}(1 - e^{-\frac{E}{\eta}t}) \qquad \textit{Kelvin model}} \tag{14.11}$$

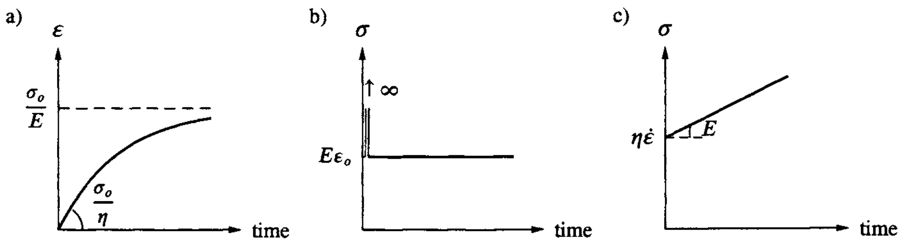

Figure 14.16: Response of Kelvin model; a) creep test, b) relaxation test and c) constant strain-rate test.

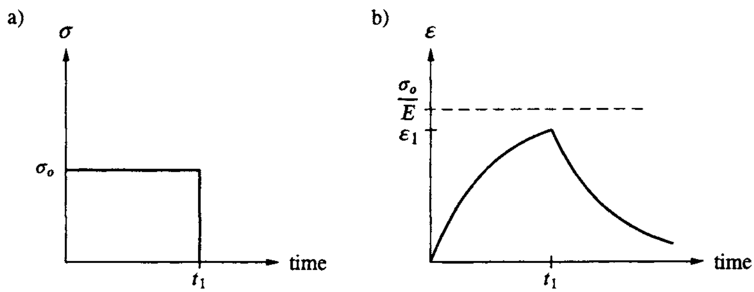

Figure 14.17: Response of Kelvin model; full recovery.

It appears from Fig. 14.16a) that the Kelvin model exhibits primary creep only. Moreover, due to the dashpot, the Kelvin model reacts as a rigid material for a sudden application of the load. Due to this peculiarity, application of the Kelvin model should be performed judiciously. This special property is also responsible for the very special response when a relaxation test is performed. According to (14.10) and in order to maintain a constant strain ε_0, the stress must be $\sigma = E\varepsilon_0$. On the other hand, in order to instantaneously deform the Kelvin model to this strain value, an infinitely large stress is required. Therefore, in a relaxation test the stress increases instantaneously to infinity and immediately after, the stress will take the value $\sigma = E\varepsilon_0$; this result is illustrated in Fig. 14.16b). For a constant strain-rate test where $\varepsilon = \dot{\varepsilon}t$ with $\dot{\varepsilon}$ being constant, (14.10) immediately gives the result shown in Fig. 14.16c). It appears that for an infinitely small strain-rate, i.e. $\dot{\varepsilon} \to 0$, the Kelvin model reacts as an elastic material.

If the stress in a creep test is removed at time t_1, the Kelvin model reacts as a rigid material during the unloading process and since $\sigma = 0$ when $t > t_1$, (14.10) implies $0 = E\varepsilon + \eta\dot{\varepsilon}$ with the solution $\varepsilon = Ce^{-\frac{E}{\eta}t}$ where the arbitrary constant C is determined by the condition $\varepsilon = \varepsilon_1$ when $t = t_1$, i.e. $\varepsilon = \varepsilon_1 e^{-\frac{E}{\eta}(t-t_1)}$ when $t \geq t_1$. This response is shown in Fig. 14.17 and with $t \to \infty$ we obtain $\varepsilon \to 0$,

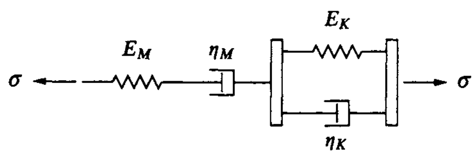

Figure 14.18: Burgers model.

i.e. the Kelvin model exhibits full recovery. Certainly, this is also evident from Fig. 14.15 since the tensile stress in the spring, when the external stress $\sigma = 0$ for $t > t_1$, will compress the dashpot until the situation $\varepsilon = 0$ has been reached.

Having discussed the simple Maxwell and Kelvin models in detail, it is evident that these models can be combined in series as shown in Fig. 14.18. We then obtain the *Burgers model* - suggested by Burgers (1935) - which represents a pretty realistic viscoelastic model. We have

$$\varepsilon = \varepsilon_M + \varepsilon_K\,; \qquad \sigma = \sigma_M = \sigma_K \tag{14.12}$$

where subscripts M and K refer to the Maxwell part and the Kelvin part, respectively. Since (14.12a) gives $\varepsilon_K = \varepsilon - \varepsilon_M$ insertion into (14.10) gives with $\sigma_K = \sigma$

$$\sigma = E_K(\varepsilon - \varepsilon_M) + \eta_K(\dot{\varepsilon} - \dot{\varepsilon}_M)$$

Differentiation with respect to time and insertion of (14.7) and using $\sigma_M = \sigma$ then give the following constitutive equation

Burgers model
$\frac{\eta_K}{E_M}\ddot{\sigma} + (1 + \frac{\eta_K}{\eta_M} + \frac{E_K}{E_M})\dot{\sigma} + \frac{E_K}{\eta_M}\sigma = \eta_K\ddot{\varepsilon} + E_K\dot{\varepsilon}$

To investigate the response in a creep test, we may force the condition $\dot{\sigma} = 0$ on the constitutive equation and this leads to a linear second order differential equation in ε, which may easily be solved. This solution will involve two arbitrary constants to be determined from the initial conditions. While this procedure is certainly possible, the identification of the initial conditions turns out to be somewhat cumbersome. Indeed, this complication occurs for all viscoelastic differential equations of a higher order than one. An easier approach is to make use of (14.3), which with (14.8) and (14.11) directly gives the result

$$\boxed{J(t) = \frac{1}{E_M} + \frac{t}{\eta_M} + \frac{1}{E_K}(1 - e^{-\frac{E_K}{\eta_K}t}) \qquad \textit{Burgers model}} \tag{14.13}$$

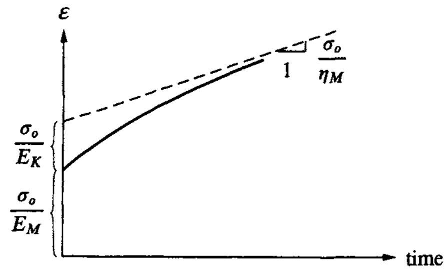

Figure 14.19: Response of Burgers model in creep test.

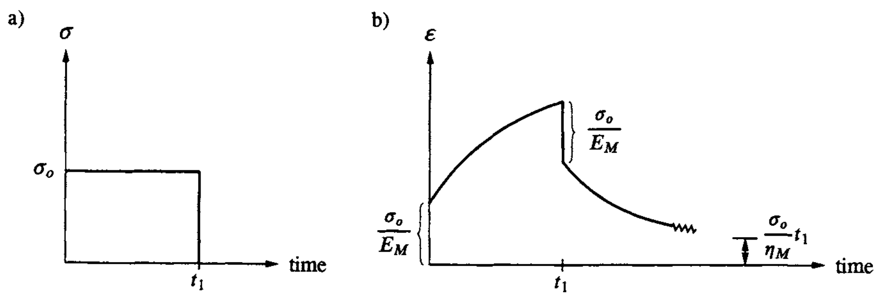

Figure 14.20: Response of Burgers model; partial recovery.

This result is shown in Fig. 14.19 and it appears that primary and - in the limit - also secondary creep occurs and that the strain rate approaches σ_0/η_M when $t \to \infty$.

If the stress in a creep test is removed at time t_1, the Maxwell part responds elastically during unloading with no further deformation when $t > t_1$, cf. Fig. 14.14, whereas the Kelvin part reacts as a rigid material during unloading and then full recovery is achieved in the Kelvin part when $t \to \infty$. Eventually, the only remaining strain is the viscous strain $\frac{\sigma_0}{\eta_M}t_1$ developed in the Maxwell part up until time t_1. Therefore, the Burgers model will show the response shown in Fig. 14.20 and only partial recovery occurs. In conclusion, Figs. 14.19 and 14.20 indicate that - except for tertiary creep - the Burgers model exhibits all the principal characteristics of a real material.

Another often used model is the *3-parameter model* shown in Fig. 14.21; this model is also called the *standard linear solid.* The constitutive equation can be obtained from (14.13) by letting $\eta_M \to \infty$. It is evident that this model will respond elastically to instantaneous loading and that it will only exhibit primary creep.

The *generalized Maxwell model* appears from Fig. 14.22. Its response in a

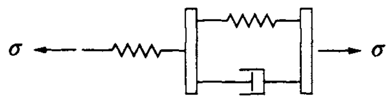

Figure 14.21: 3-parameter model (standard linear solid).

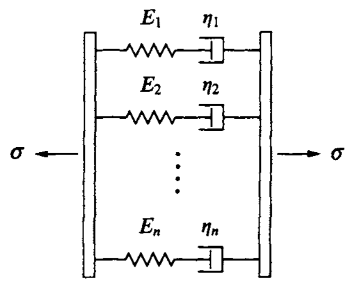

Figure 14.22: Generalized Maxwell model.

relaxation test can be inferred from (14.4) and (14.8) to provide

Generalized Maxwell model

$$G(t) = \sum_{i=1}^{n} E_i e^{-\frac{E_i}{\eta_i}t} \tag{14.14}$$

It is evident that a close approximation can be obtained to experimental data. In a creep test it can be shown that the response will also involve primary creep. In general, the response is controlled by the following equations

$$\varepsilon = \varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n\,; \qquad \sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_n$$

and from (14.7) follows that

$$\begin{aligned}
\dot{\varepsilon}_1 &= \frac{\dot{\sigma}_1}{E_1} + \frac{\sigma_1}{\eta_1} \\
\dot{\varepsilon}_2 &= \frac{\dot{\sigma}_2}{E_2} + \frac{\sigma_2}{\eta_2} \\
&\vdots \\
\dot{\varepsilon}_n &= \frac{\dot{\sigma}_n}{E_n} + \frac{\sigma_n}{\eta_n}
\end{aligned}$$

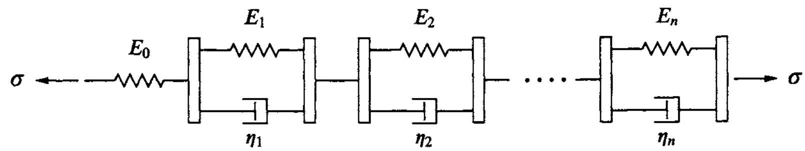

Figure 14.23: Generalized Kelvin model.

It is easily checked that there are $2n + 1$ of these constitutive equations which involve the $2 + 2n$ unknowns given by $\varepsilon, \sigma, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n, \sigma_1, \sigma_2, \cdots, \sigma_n$. By a tedious elimination process, the result becomes one constitutive relation in terms of one higher-order differential equation in ε and σ.

The *generalized Kelvin model* is given in Fig. 14.23 where the elastic spring with the stiffness E_0 has been added in order that the response to an instantaneous loading be elastic. Its response in a creep test can be inferred from (14.3) and (14.11) to provide

Generalized Kelvin model

$$\boxed{J(t) = \frac{1}{E_0} + \sum_{i=1}^{n} \frac{1}{E_i}(1 - e^{-\frac{E_i}{\eta_i}t})} \tag{14.15}$$

It appears that primary creep, only, can be modeled even though a large possibility exists for close approximations to experimental data; possible secondary creep can be modeled by adding a dashpot in series with the spring E_0. The series in (14.15) containing exponentials is called a *Dirichlet series* or - occasionally - a *Prony series.* In general, the response of the model is given by the following constitutive equations

$$\varepsilon = \varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n\,; \qquad \sigma = \sigma_0 = \sigma_1 = \sigma_2 = \cdots = \sigma_n$$

and for the elastic spring and from (14.10) follow that

$$\begin{aligned}
\sigma_0 &= E_0\varepsilon_0 \\
\sigma_1 &= E_1\varepsilon_1 + \eta_1\dot{\varepsilon}_1 \\
\sigma_2 &= E_2\varepsilon_2 + \eta_2\dot{\varepsilon}_2 \\
&\vdots \\
\sigma_n &= E_n\varepsilon_n + \eta_n\dot{\varepsilon}_n
\end{aligned}$$

Again a tedious elimination process makes it possible to obtain one constitutive relation in terms of a higher-order differential equation in time of ε and σ.

Certainly, a very comprehensive and accurate model may be constructed by combining a generalized Maxwell model and a generalized Kelvin model. Irrespective of how we combine springs and dashpots, it comes as no surprise that it is always possible to write the constitutive equation in the following general form

General format of the constitutive equation

$$\sum_{i=0}^{n} a_i \frac{d^i\sigma}{dt^i} = \sum_{j=1}^{m} b_j \frac{d^j\varepsilon}{dt^j} \tag{14.16}$$

where, per definition, we have $\frac{d^0\sigma}{dt^0} = \sigma$ and $\frac{d^0\varepsilon}{dt^0} = \varepsilon$. In (14.16), a_i and b_j are some constant material parameters and, in general, n and m differ. An advantage of the differential equation approach discussed above is that each model is easy to physically understand and interpret and models can be constructed in an intuitive fashion. The drawback, however, is that the more advanced models soon become cumbersome to deal with and as already touched on in relation to the Burgers model the implication of a higher-order differential equation in time is that the initial conditions become difficult to deal with.

We have seen that an exponential term in the form $e^{-\frac{t}{t_r}}$, where t_r is a constant with the dimension of time, often emerges, cf. (14.8) and (14.11). It appears that this exponential term is unity for $t = 0$ whereas it has decreased to the value $\frac{1}{e}$ when $t = t_r$. If t_r emerges in the creep compliance function, cf. (14.11) where $t_r = \frac{\eta}{E}$, t_r is called a *retardation time* and if it emerges in the relaxation modulus, cf. (14.8) where t_r again takes the value $\frac{\eta}{E}$, it is called a *relaxation time*. If the creep compliance $J(t)$ or relaxation modulus $G(t)$ contain more exponential terms, cf. (14.15) and (14.14), it is possible to speak of a retardation or relaxation *spectrum*. In that case, a good approximation to experimental data over a large time span can be achieved if the retardation - or relaxation - times are spread uniformly; typically, a factor of 10 is chosen between these times, cf. for instance Bažant (1982) and Mills (1986).

Returning to the constitutive relation (14.16), it appears that this differential equation is a linear and the *superposition principle* then holds - as expected. Therefore, if the stress history $\sigma = \sigma_1(t)$ implies the response $\varepsilon = \varepsilon_1(t)$, then the stress history $\sigma = k\sigma_1(t)$, where k is a constant, implies the response $\varepsilon = k\varepsilon_1(t)$. Moreover, if the stress history $\sigma = \sigma_1(t)$ implies $\varepsilon = \varepsilon_1(t)$ and the stress history $\sigma = \sigma_2(t)$ implies $\varepsilon = \varepsilon_2(t)$ then the combined stress history $\sigma = \sigma_1(t) + \sigma_2(t)$ implies $\varepsilon = \varepsilon_1(t) + \varepsilon_2(t)$.

In (14.16), the material parameters a_i and b_j are constants, but we can relax this requirement and still maintain the superposition principle; as long as a_i and b_j do not depend on the stress or strain, the superposition principle will still hold. As an example, creep depends strongly on temperature and we may then let $a_i = a_i(T)$ and $b_j = b_j(T)$ where T is the temperature.

It turns out that the effect of temperature can often be evaluated in a simple and elegant fashion and for this purpose consider the relaxation modulus defined by (14.2). Since the material parameters now depend on the temperature T we have $G = G(t, T)$. Consider now relaxation tests performed at different temperatures where T_0 is a reference temperature. Since increase of temperature enhances creep deformation and plotting experimental data against logarithmic time, the results shown in Fig. 14.24a) are obtained.

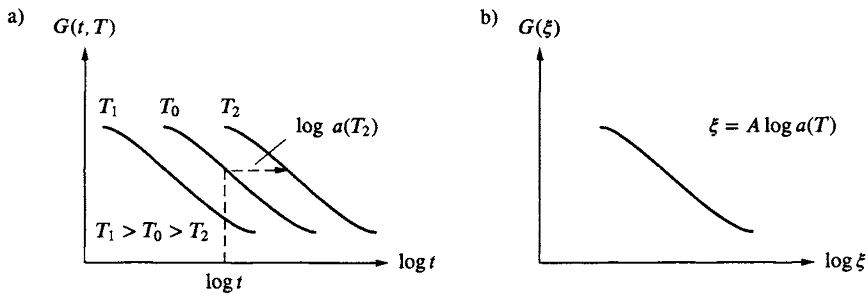

Figure 14.24: Relaxation data plotted against logarithmic times; a) original data, b) master relaxation curve.

For polymers, cf. Findley *et al.* (1976), and also for concrete, see Mukaddam and Bresler (1972), it is often observed experimentally that if the curves in Fig. 14.24a) are moved horizontally they can be brought to coincide; therefore, if the relaxation curve for the reference temperature T_0 is moved the horizontal distance $a(T_2)$, where $a(T)$ is a function of the temperature, then this curve coincides with the relaxation curve for the temperature T_2, cf. Fig. 14.24a). With this property we have

$$G_{T_0}(\log t) = G_{T_2}(\log t + \log a(T_2))$$

This implies that if the *reduced time* ξ is defined by $\xi = t\,a(T)$, where $a(T_0) = 1$, then all relaxation moduli can be written as $G(t,T) = G(\xi)$ and all curves in Fig. 14.24a) can then be expressed by one curve as shown in Fig. 14.24b); this curve is called the *master relaxation curve*. The procedure above is called the *time-temperature shift principle* and evidently the same concepts hold for the creep compliance, i.e.

Time-temperature shift principle

$$G(t,T) = G(\xi)\,; \qquad J(t,T) = J(\xi)$$

The reduced time ξ is defined by

$$\xi = t\,a(T)$$

where $a(T)$ is the shift-factor and $a(T_0) = 1$

In reality, this principle is not a principle *per se*, but it is an elegant procedure that can be adapted to many materials; it was introduced by Alfrey (1957) and Schwarzl and Staverman (1952) and expressions for the shift-factor $a(T)$ can be found, for instance, in Findley *et al.* (1976). If the temperature varies, it seems reasonable to use $\xi = \int_0^t a(T(\tau))d\tau$ where $T(\tau)$ expresses the temperature - time history.

Returning to (14.16) and in order to achieve a better correlation with experimental data, the material parameters are sometimes allowed to vary with the loading time. Since the material parameters then change - or *harden* - with time, such models are called *time-hardening models*; still the superposition principle holds. As an example, let the viscosity parameter η in the Maxwell model depend on the loading time, as suggested by Dischinger (1937). Then the constitutive relation (14.7) becomes

Dischinger model

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta(t)} \tag{14.17}$$

exhibits time-hardening

The *Dischinger model* is very often used for modeling creep of concrete and by choosing a proper function for $\eta(t)$ close agreement with experimental data can be obtained; this model forms the basis in many national design codes. However, time-hardening models should be used with care and to illustrate that Fig. 14.25 is considered. In Fig. 14.25a), the stress is infinitely small up to $t = t_1$ where it is increased instantaneously to σ_0; since the loading time starts at time $t = 0$ the strain rate at $t = t_1$ according to (14.17) is $\dot{\varepsilon} = \sigma_0/\eta(t_1)$. In Fig. 14.25b), the loading up to time t_1 is now exactly zero and since the loading time therefore starts at time t_1 the strain rate following (14.17) becomes $\dot{\varepsilon} = \sigma_0/\eta(0)$. In reality, the two responses shown in Fig. 14.25a) and b) must be identical and this illustrates the problems with using loading time as a *creep hardening parameter*; we will return to this problem later.

Apart from creep hardening, another issue is that of *aging*, i.e. the material parameters change with time irrespective of the material being loaded or not. This aging effect is prominent when, for instance, glue is setting or concrete is hardening after it has been cast. Since such aging effects are strongly dependent on the temperature, it is a poor measure for aging just to use time as a parameter; some kind of *maturity concept* is more realistic. For hardening concrete, for instance, one may define an *equivalent maturity time* t_e such that t_e is the same for two concrete specimens (made of the same composition) if the aging is the same despite the two specimens having been exposed to different temperature-time histories. This maturity concept was introduced by Plowman (1956) and a review is given by Byfors (1980). The equivalent maturity time t_e is defined by

Equivalent maturity time t_e for measuring aging effects

$$t_e = \int_0^t \frac{f(\theta(\tau))}{f(\theta_0)}\, d\tau \tag{14.18}$$

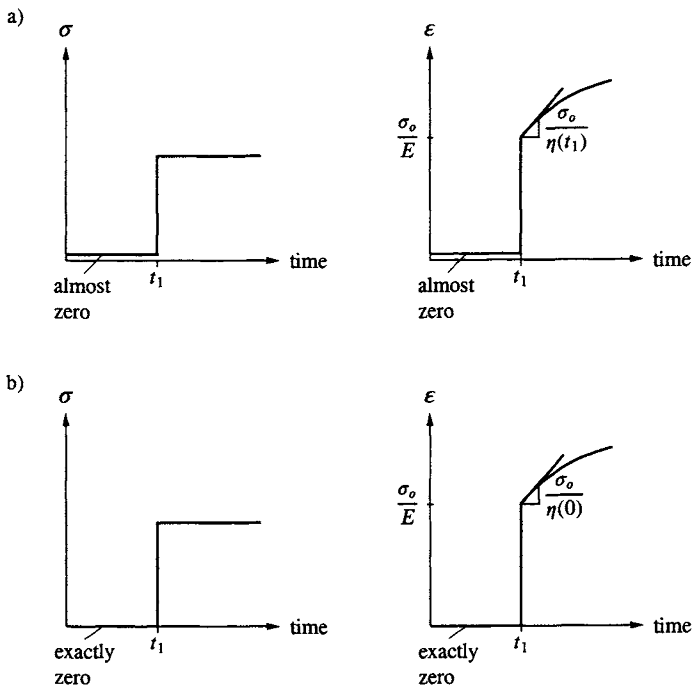

Figure 14.25: Creep tests of Dischinger's time-hardening model; a) between $t = 0$ and $t = t_1$ the stress is infinitely small, b) between $t = 0$ and $t = t_1$ the stress is exactly zero.

where f is a function of the absolute temperature θ $[K]$ and θ_0 is a reference temperature. In (14.18) $\theta = \theta(\tau)$ expresses the temperature-time history and the integration above is performed from time zero where the concrete was cast up until the current time t. As function $f(\theta)$, the *Arrhenius function* for thermal activation is often adopted, i.e. $f(\theta) = ke^{-\frac{Q}{R\theta}}$ where k is a constant, Q is the *activation energy for creep* $[J/mol]$ for the material in question and R is the *universal gas constant* = 8.314 $[J/mol\ K]$.

In view of these remarks, it then seems reasonable to model creep of aging materials by letting the material parameters in (14.16) depend on the equivalent maturity time t_e, i.e. $a_i = a_i(t_e)$ and $b_j = b_j(t_e)$.

Nonlinear viscoelasticity based on (14.16) can be achieved by letting the parameters a_i and b_i depend on the stress and/or strain and we will return to this important possibility in Section 15.3.

With this detailed discussion of various linear viscoelastic models exposed to uniaxial stress conditions, it is timely to see how these models can be generalized to three-dimensional stress conditions. Indeed this is straightforward and from (14.16) it follows directly that

General format for three-dimensional loading

$$\sum_{\alpha=0}^{n} A^{\alpha}_{ijkl} \frac{d^{\alpha}\sigma_{kl}}{dt^{\alpha}} = \sum_{\beta=0}^{m} B^{\beta}_{ijkl} \frac{d^{\beta}\varepsilon_{kl}}{dt^{\beta}}$$

which in a matrix format reads

$$\sum_{\alpha=0}^{n} \boldsymbol{A}^{\alpha} \frac{d^{\alpha}\boldsymbol{\sigma}}{dt^{\alpha}} = \sum_{\beta=0}^{m} \boldsymbol{B}^{\beta} \frac{d^{\beta}\boldsymbol{\varepsilon}}{dt^{\beta}} \tag{14.19}$$

where A^{α}_{ijkl} and B^{β}_{ijkl} are fourth-order tensors; the corresponding matrices $\boldsymbol{A}^{\alpha}$ and $\boldsymbol{B}^{\beta}$ are obtained in a fashion similar to the discussion in Section 4.4.

As an example, consider a Maxwell model for isotropic material behavior. A comparison with (14.7) shows that

$$A^{1}_{ijkl}\dot{\sigma}_{kl} + A^{0}_{ijkl}\sigma_{kl} = B^{1}_{ijkl}\dot{\varepsilon}_{kl} \tag{14.20}$$

With A^{1}_{ijkl} and A^{0}_{ijkl} being isotropic fourth-order tensors, cf. (4.93), and B^{1}_{ijkl} being the *unit fourth-order tensor*, i.e. $B^{1}_{ijkl}\dot{\varepsilon}_{kl} = \dot{\varepsilon}_{ij}$, we obtain

$$\begin{aligned} A^{1}_{ijkl} &= C_{ijkl} = \frac{1}{2G}[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\nu}{1+\nu}\delta_{ij}\delta_{kl}] \\ A^{0}_{ijkl} &= \frac{1+\xi}{\eta}\frac{1}{2}[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\xi}{1+\xi}\delta_{ij}\delta_{kl}] \\ B^{1}_{ijkl} &= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned} \tag{14.21}$$

where G and ν as usual denote the shear modulus and Poisson's ratio, respectively, whereas η is a viscosity material parameter and ξ is a dimensionless material parameter; moreover, A^{1}_{ijkl} is recognized as the isotropic elastic flexibility tensor C_{ijkl}. Insertion of (14.21) into (14.20) provides

Three-dimensional isotropic Maxwell model

$$\frac{1}{2G}(\dot{\sigma}_{ij} - \frac{\nu}{1+\nu}\delta_{ij}\dot{\sigma}_{kk}) + \frac{1+\xi}{\eta}(\sigma_{ij} - \frac{\xi}{1+\xi}\delta_{ij}\sigma_{kk}) = \dot{\varepsilon}_{ij} \tag{14.22}$$

Apart from a redefinition of parameters, this expression corresponds exactly to (6.49), which, however, was derived by somewhat different means.

In accordance with (14.6), (14.22) may be written as $\dot{\varepsilon}^{e}_{ij} + \dot{\varepsilon}^{v}_{ij} = \dot{\varepsilon}_{ij}$ and it is concluded that the viscous strain rate is given by $\dot{\varepsilon}^{v}_{ij} = \frac{1+\xi}{\eta}(\sigma_{ij} - \frac{\xi}{1+\xi}\delta_{ij}\sigma_{kk})$ which for the choice $\xi = 1/2$ gives $\dot{\varepsilon}^{v}_{ij} = \frac{3}{2\eta}s_{ij}$, i.e. the viscous strains - or the creep strains - are incompressible if the material parameter ξ is chosen as $\xi = 1/2$ and the volume changes are then purely elastic.

Another choice of the material parameter ξ becomes apparent if (14.22) is specialized to uniaxial stress conditions. Evaluating the axial strain rate $\dot{\varepsilon}_{11}$ and the transverse strain rate $\dot{\varepsilon}_{22} = \dot{\varepsilon}_{33}$, we obtain

$$\begin{aligned} \frac{1}{E}\dot{\sigma}_{11} + \frac{1}{\eta}\sigma_{11} &= \dot{\varepsilon}_{11} \\ -\frac{\nu}{E}\dot{\sigma}_{11} - \frac{\xi}{\eta}\sigma_{11} &= \dot{\varepsilon}_{22} \end{aligned} \tag{14.23}$$

The first of these equations is in correspondence with (14.7). However, if ξ is chosen as $\xi = \nu$ then (14.23) shows that the relation $\dot{\varepsilon}_{22} = -\nu\dot{\varepsilon}_{11}$ holds not only for elastic conditions, but also during development of creep strains. This convenient situation is often supported experimentally for viscoelastic materials, for instance for concrete, cf. Hannant (1969) and Browne and Blundell (1972).

As another example, consider a Kelvin model for isotropic material behavior. A comparison of (14.19) with (14.10) shows that

$$A^0_{ijkl}\sigma_{kl} = B^0_{ijkl}\varepsilon_{kl} + B^1_{ijkl}\dot{\varepsilon}_{kl} \tag{14.24}$$

With A^0_{ijkl} being the unit fourth-order tensor whereas B^0_{ijkl} and B^1_{ijkl} are isotropic fourth-order tensors, cf. (4.89), we obtain

$$\begin{aligned} A^0_{ijkl} &= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ B^0_{ijkl} &= D_{ijkl} = 2G[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}] \\ B^1_{ijkl} &= \frac{\eta}{1+\nu}[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\xi}{1-2\xi}\delta_{ij}\delta_{kl}] \end{aligned} \tag{14.25}$$

As before, G and ν denote the shear modulus and Poisson's ratio, respectively, whereas η is a viscosity material parameter and ξ is a dimensionless material parameter; moreover, B^0_{ijkl} is recognized as the isotropic elastic stiffness tensor D_{ijkl}. Intuitively, it seems more natural to choose the common factor in B^1_{ijkl} as $\frac{\eta}{1+\xi}$, but the choice $\frac{\eta}{1+\nu}$ turns out to be more convenient as the later calculations will show. Insertion of (14.25) into (14.24) gives

Three-dimensional isotropic Kelvin model

$$\boxed{\sigma_{ij} = 2G(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}) + \frac{\eta}{1+\nu}(\dot{\varepsilon}_{ij} + \frac{\xi}{1-2\xi}\delta_{ij}\dot{\varepsilon}_{kk})} \tag{14.26}$$

Apart from a redefinition of parameters, this expression corresponds exactly to (6.56), which, however, was derived by somewhat different means. The constitutive relation above implies $\sigma_{ii} = 3K\varepsilon_{ii} + \frac{\eta}{1+\nu}\frac{1+\xi}{1-2\xi}\dot{\varepsilon}_{ii}$ and the choice $\xi = -1$ therefore results in $\sigma_{ii} = 3K\varepsilon_{ii}$, i.e. the volumetric response is purely elastic.

Another choice of the material parameter ξ becomes apparent if (14.26) is specialized to uniaxial stress conditions where ε_{11} is the axial strain and $\varepsilon_{22} = \varepsilon_{33}$ is the transverse strain. If we again enforce the condition that $\varepsilon_{22} = -\nu\varepsilon_{11}$ should hold even during creep development, we obtain $\varepsilon_{kk} = (1-2\nu)\varepsilon_{11}$ and (14.26) evaluated for $ij = 11$ and $ij = 22$ then provides

$$\begin{aligned}\sigma_{11} &= E\varepsilon_{11} + \frac{\eta}{1+\nu}(1+\xi\frac{1-2\nu}{1-2\xi})\dot{\varepsilon}_{11} \\ 0 &= \frac{\eta}{1+\nu}\frac{\xi-\nu}{1-2\xi}\dot{\varepsilon}_{11}\end{aligned}$$

The second of these equations is fulfilled for $\xi = \nu$ and the first equation then reduces to

$$\sigma_{11} = E\varepsilon_{11} + \eta\dot{\varepsilon}_{11}$$

which corresponds to (14.10). We have then shown that the choice $\xi = \nu$ implies that $\varepsilon_{22} = -\nu\varepsilon_{11}$ holds during uniaxial stress conditions.

Above we have considered isotropic material behavior, but if the material is, say, orthotropic, it is often more convenient to work with the matrix version of (14.19). Then the matrices $\boldsymbol{A}^{\alpha}$ and $\boldsymbol{B}^{\beta}$ are either unit matrices or orthotropic matrices and the orthotropic matrices will each contain nine independent material parameters in complete similarity with the discussion in Section 4.6, cf. in particular the orthotropic matrix format given by (4.55). Orthotropic viscoelasticity is often used to model creep of wood, see Mårtensson (1992) and Ormarsson (1999), and paper, see Lif *et al.* (1999).

14.3 Hereditary approach

We will now introduce the *hereditary approach* to linear viscoelasticity, which allows greater freedom when constructing models than the differential approach that relies on the concepts of certain combinations of springs and dashpots. The essential issue in the hereditary approach is that of superposition.

Consider the uniaxial stress history in Fig. 14.26a) where the stress is increased instantaneously the constant amount $\Delta\sigma$ at time τ. If $\Delta\sigma$ were applied at time $t = 0$, the corresponding strain at time t would be given by $\Delta\varepsilon(t) = J(t)\Delta\sigma$, cf. (14.1), but now $\Delta\sigma$ is applied at time τ so the strain $\Delta\varepsilon(t)$ at time t caused by $\Delta\sigma$ applied at time τ becomes

$$\Delta\varepsilon(t) = J(t-\tau)\Delta\sigma$$

For an infinitesimal stress change $d\sigma$ applied at time τ, we then obtain $d\varepsilon(t) = J(t-\tau)d\sigma$ and integration all infinitesimal stress changes over the entire load history up until the current time t then provides

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)d\sigma(\tau) \tag{14.27}$$

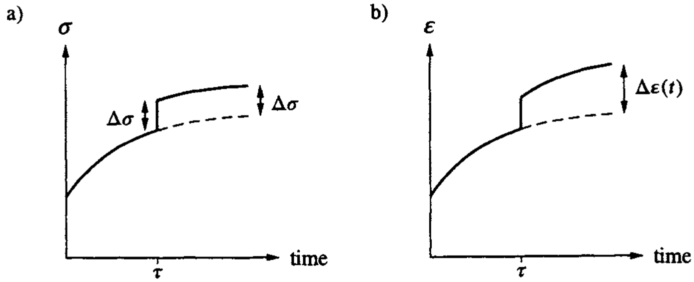

Figure 14.26: a) Stress history where the stress is increased the constant amount $\Delta\sigma$ at time τ, b) corresponding strain response.

where the notation $d\sigma(\tau)$ expresses that the infinitesimal stress changes are given as a function of the stress history $\sigma = \sigma(\tau)$. In (14.27), the lower integration limit is by tradition taken as $-\infty$, but since the stress is equal to zero up until time zero, where the loading begins, the contribution form the integral in (14.27) from $-\infty$ to zero is nil. Moreover, suppose that an instantaneous loading occurs at time zero according to

$$\sigma(t) = \begin{cases} 0 & \text{when} \;\; t < 0 \\ \sigma_0 + \sigma_1(t)\,; \quad \sigma_1(0) = 0 & \text{when} \;\; t \geq 0 \end{cases}$$

then care should be taken in the integration. With evident notation we obtain

$$\begin{aligned} \varepsilon(t) &= \int_{-\infty}^{0^-} J(t-\tau)d\sigma(\tau) + \int_{0^-}^{0^+} J(t-\tau)d\sigma(\tau) + \int_{0^+}^{t} J(t-\tau)d\sigma(\tau) \\ &= 0 + J(t)\sigma_0 + \int_{0^+}^{t} J(t-\tau)d\sigma(\tau) \end{aligned}$$

i.e.

$$\varepsilon(t) = J(t)\sigma_0 + \int_{0^+}^{t} J(t-\tau)d\sigma(\tau)$$

With this interpretation, it appears that jumps are allowed in the stress history and the integral in (14.27) is therefore a so-called *Stieltjes integral.*

However, if the stress history $\sigma = \sigma(\tau)$ is smooth, we have $d\sigma(\tau) = \frac{d\sigma(\tau)}{d\tau}d\tau$ and (14.27) then takes the form

Hereditary approach

$$\boxed{\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau} \qquad (14.28)$$

where the integral is the usual *Rieman integral.* However, in the following we will adopt the format (14.28) in the sense that if the stress history exhibits jumps, then the interpretation (14.27) should be used. If the creep compliance $J(t)$ is known then for any given stress history, (14.28) provides the corresponding strain. Since the current strain $\varepsilon(t)$ is obtained as an integration over the entire loading history, the terminology of *hereditary approach* is assigned to the format (14.28). Moreover, the integral is an example of a so-called *convolution integral* and occasionally it is also called a *Duhamel integral.*

The result (14.28) hinges only on the superposition principle which is called *Boltzmann's superposition principle* and is due to Boltzmann (1874), but the specific format given by (14.28) is due to Volterra (1913) and a material obeying (14.28) is therefore also called a *Boltzmann-Volterra material.* In (14.28), knowledge of $J(t-\tau)$ and of the stress history provides the corresponding strain. However, if in (14.28) the strain $\varepsilon(t)$ and the creep compliance $J(t-\tau)$ are taken as given and the stress history $\sigma = \sigma(\tau)$ is taken as unknown, (14.28) represents an *integral equation* which, not surprisingly, turns out to be a *Volterra integral equation.* Viewed in this manner $J(t-\tau)$ comprises the so-called *kernel* and for more details see, for instance, Hildebrand (1965) and Volterra (1959).

Since (14.28) only rests on the superposition principle, the models derived in the previous section can be recast into this format. As an example, take the Maxwell model with the creep compliance given by (14.8). From (14.28) it then follows that

$$
\begin{aligned}
\varepsilon(t) &= \int_{-\infty}^{t} [\frac{1}{E} + \frac{t-\tau}{\eta}] \frac{d\sigma(\tau)}{d\tau} d\tau \\
&= (\frac{1}{E} + \frac{t}{\eta}) \int_{0^-}^{t} d\sigma(\tau) - \frac{1}{\eta} \int_{0^-}^{t} \tau \, d\sigma(\tau) \\
&= (\frac{1}{E} + \frac{t}{\eta})\sigma - \frac{1}{\eta}\{[\tau\, \sigma(\tau)]_{0^-}^{t} - \int_{0^-}^{t} \sigma(\tau)\, d\tau\} \\
&= \frac{\sigma}{E} + \frac{1}{\eta} \int_{0^-}^{t} \sigma(\tau)\, d\tau
\end{aligned}
$$

Differention with respect to time then provides

$$
\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}
$$

in accordance with (14.7).

Therefore, per definition, the format (14.28) includes the models derived in the previous section, but (14.28) is more general since we can now specify any creep compliance $J(t)$ and for a given stress history, (14.28) determines the current strain. Therefore, the creep compliance $J(t)$ ca be identified entirely by means of experimental evidence and no resort has to be taken to an interpretation in terms of springs and dashpots.

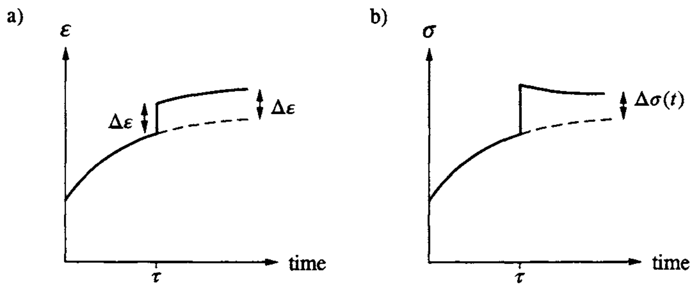

Figure 14.27: a) Strain history where the strain is increased the constant amount $\Delta\varepsilon$ at time τ, b) corresponding stress response.

Let us now assume that the strain history is known and let us derive a result analogous to (14.28). If the constant strain change $\Delta\varepsilon$ is applied at time τ, see Fig. 14.27a), the corresponding stress change $\Delta\sigma(t)$ at the current time t is given by $\Delta\sigma(t) = G(t-\tau)\Delta\varepsilon$, cf. (14.2). We are then led to

$$\sigma(t) = \int_{-\infty}^{t} G(t-\tau)\, d\varepsilon(\tau) \tag{14.29}$$

where the notation $d\varepsilon(\tau)$ expresses that the infinitesimal strain changes are given as function of the strain history $\varepsilon = \varepsilon(\tau)$. If a jump exists in the strain history, the interpretation of (14.29) is similar to (14.27), i.e. (14.29) is a Stieltjes integral. If the strain history $\varepsilon = \varepsilon(\tau)$ is smooth, (14.29) takes the format

Hereditary approach

$$\sigma(\tau) = \int_{-\infty}^{t} G(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}\, d\tau \tag{14.30}$$

Again the lower integration limit is per tradition taken as $-\infty$, but the integral contributes with nil up to time zero where the strain is applied, cf. the similar discussion following (14.27). It appears that once the relaxation modulus $G(t)$ and the strain history are known, (14.30) provides the corresponding stress.

Since the format (14.30) only relies on the superposition principle, this format contains all the models discussed in the previous section. As an example, take the Maxwell model with the relaxation modulus $G(t)$ given by (14.8). From (14.30) it then follows that

$$\begin{aligned}\sigma(t) &= \int_{-\infty}^{t} Ee^{-\frac{E}{\eta}(t-\tau)}\frac{d\varepsilon(\tau)}{d\tau}d\tau \\ &= Ee^{-\frac{E}{\eta}t}\int_{0^-}^{t} e^{\frac{E}{\eta}\tau}\frac{d\varepsilon(\tau)}{d\tau}d\tau\end{aligned}$$

Differentiation with respect to time gives

$$\dot{\sigma} = -\frac{E^2}{\eta} e^{-\frac{E}{\eta}t} \int_{0^-}^{t} e^{\frac{E}{\eta}\tau} \frac{d\varepsilon(\tau)}{d\tau} d\tau + E\dot{\varepsilon}$$

and elimination of the integral term by means of the first equation results in

$$\dot{\sigma} = -\frac{E}{\eta}\sigma + E\dot{\varepsilon}$$

in accordance with (14.7).

However, the format (14.30) is more general than the format discussed in the previous section since it is now possible from experimental evidence to directly propose any relaxation modulus $G(t)$ and for a given strain history, (14.30) then provides the corresponding stress.

We have previously indicated that one might expect that some relation exists between the creep compliance $J(t)$ and the relaxation modulus $G(t)$; let us now identify this relation.

Consider a creep test where the constant stress σ_0 is applied instantaneously at time $t = 0$. Then (14.28) - or (14.27) - gives

$$\varepsilon(t) = J(t)\sigma_0$$

as expected. Supposing that this strain history is known, then insertion into (14.30) should provide $\sigma(t) = \sigma_0$, i.e.

$$\boxed{\int_{-\infty}^{t} G(t-\tau)\frac{dJ(\tau)}{d\tau} d\tau = 1} \tag{14.31}$$

In the integration, it is noted that both G and J are zero when $t < 0$ and the integration for $-\infty$ up to $t = 0^-$ therefore gives no contribution and most often there will be a discontinuity in $J(t)$ at time $t = 0$.

Consider next a relaxation test where the constant strain ε_0 is applied instantaneously at time $t = 0$. Then (14.30) - or (14.29) - gives

$$\sigma(t) = G(t)\varepsilon_0$$

as expected. Supposing that this stress history is known, then insertion into (14.28) should provide $\varepsilon(t) = \varepsilon_0$, i.e.

$$\boxed{\int_{-\infty}^{t} J(t-\tau)\frac{dG(\tau)}{d\tau} d\tau = 1}$$

In this integration similar arguments hold to those discussed in relation to (14.31).

The freedom with which we can choose $J(t)$ - or $G(t)$ - is a major advantage of the hereditary approach. Moreover, it is easy to generalize to three-dimensional loading and we obtain directly for (14.28) and (14.30)

> *Three-dimensional loading*
>
> $$\varepsilon_{ij}(t) = \int_{-\infty}^{t} J_{ijkl}(t-\tau)\frac{d\sigma_{kl}(\tau)}{d\tau}d\tau$$
>
> *or*
>
> $$\sigma_{ij}(t) = \int_{-\infty}^{t} G_{ijkl}(t-\tau)\frac{d\varepsilon_{kl}(\tau)}{d\tau}d\tau$$

which in evident matrix notation becomes

$$\begin{aligned}\boldsymbol{\varepsilon}(t) &= \int_{-\infty}^{t} \boldsymbol{J}(t-\tau)\frac{d\boldsymbol{\sigma}(\tau)}{d\tau}d\tau \\ \boldsymbol{\sigma}(t) &= \int_{-\infty}^{t} \boldsymbol{G}(t-\tau)\frac{d\boldsymbol{\varepsilon}(\tau)}{d\tau}d\tau\end{aligned} \tag{14.32}$$

We will now illustrate a particular property relating to the hereditary approach. Suppose that $\boldsymbol{J}(t-\tau)$ and the stress history $\boldsymbol{\sigma}(\tau)$ are known; at time $t = t_1$, (14.32a) then gives

$$\boldsymbol{\varepsilon}(t_1) = \int_{-\infty}^{t_1} \boldsymbol{J}(t_1-\tau)\frac{d\boldsymbol{\sigma}(\tau)}{d\tau}d\tau$$

Consider now the strains at time $t = t_1 + \Delta t$ where Δt is a small time increment; we obtain

$$\boldsymbol{\varepsilon}(t_1+\Delta t) = \int_{-\infty}^{t_1+\Delta t} \boldsymbol{J}(t_1+\Delta t-\tau)\frac{d\boldsymbol{\sigma}(\tau)}{d\tau}d\tau$$

i.e.

$$\begin{aligned}\boldsymbol{\varepsilon}(t_1+\Delta t) = &\int_{-\infty}^{t_1} \boldsymbol{J}(t_1+\Delta t-\tau)\frac{d\boldsymbol{\sigma}(\tau)}{d\tau}d\tau \\ &+ \int_{t_1}^{t_1+\Delta t} \boldsymbol{J}(t_1+\Delta t-\tau)\frac{d\boldsymbol{\sigma}(\tau)}{d\tau}d\tau\end{aligned}$$

The important thing is that the first term on the right-hand side is not equal to $\boldsymbol{\varepsilon}(t_1)$ since the argument in the function $\boldsymbol{J}$ differs in the two cases. Therefore the first term on the right-hand side needs to be integrated from the beginning of the load history. Evidently, this complicates the application since at each time one must perform an integration over the entire load history to obtain the corresponding strain; indeed this is a consequence of the hereditary approach where the current response is a result of the entire load history. However, this complicates the (numerical) determination of the response and it may be argued that the response of a material is more dependent on its recent history than on its past; this expectation is referred to as the concept of *fading memory* which is often adopted in constitutive mechanics, cf. the discussion given by Eringen (1975b); let us now see how the above problems can be circumvented.

The generalized Kelvin model appears from Fig. 14.23 and it was shown that close predictions to experimental data can be obtained with this model. The corresponding creep compliance is given by (14.15), i.e

$$J(t) = \frac{1}{E_0} + \sum_{i=1}^{n} \frac{1}{E_i}(1 - e^{-\frac{E_i}{\eta_i}t})$$

This expression is immediately generalized to three-dimensional loading according to

$$\boldsymbol{J}(t) = \boldsymbol{C}_0 + \sum_{i=1}^{n} \boldsymbol{C}_i(1 - e^{-\frac{E_i}{\eta_i}t})$$

where $\boldsymbol{C}_0$ and $\boldsymbol{C}_i$ are constant matrices. Insertion into (14.32) provides

$$\boldsymbol{\varepsilon}(t) = \int_{-\infty}^{t} [\boldsymbol{C}_0 + \sum_{i=1}^{n} \boldsymbol{C}_i(1 - e^{-\frac{E_i}{\eta_i}(t-\tau)})]\frac{d\boldsymbol{\sigma}(\tau)}{d\tau}d\tau$$

which can be written as

$$\boxed{\begin{aligned} &\textit{Generalized Kelvin} \\ &\boldsymbol{\varepsilon}(t) = (\boldsymbol{C}_0 + \sum_{i=1}^{n} \boldsymbol{C}_i)\boldsymbol{\sigma}(t) - \sum_{i=1}^{n} e^{-\frac{E_i}{\eta_i}t}\tilde{\boldsymbol{\varepsilon}}_i(t) \\ &\textit{where} \\ &\tilde{\boldsymbol{\varepsilon}}_i(t) = \boldsymbol{C}_i \int_{-\infty}^{t} e^{\frac{E_i}{\eta_i}\tau}\frac{d\boldsymbol{\sigma}(\tau)}{d\tau}d\tau \end{aligned}} \tag{14.33}$$

At time $t = t_1$, we obtain

$$\boldsymbol{\varepsilon}(t_1) = (\boldsymbol{C}_0 + \sum_{i=1}^{n} \boldsymbol{C}_i)\boldsymbol{\sigma}(t_1) - \sum_{i=1}^{n} e^{-\frac{E_i}{\eta_i}t_1}\tilde{\boldsymbol{\varepsilon}}_i(t_1) \tag{14.34}$$

and at time $t = t_1 + \Delta t$ where Δt is a small time increment, we have with $\boldsymbol{\sigma}(t_1 + \Delta t) = \boldsymbol{\sigma}(t_1) + \Delta\boldsymbol{\sigma}$

$$\begin{aligned} \boldsymbol{\varepsilon}(t_1 + \Delta t) =& (\boldsymbol{C}_0 + \sum_{i=1}^{n} \boldsymbol{C}_i)(\boldsymbol{\sigma}(t_1) + \Delta\boldsymbol{\sigma}) \\ &- \sum_{i=1}^{n} e^{-\frac{E_i}{\eta_i}(t_1+\Delta t)}\tilde{\boldsymbol{\varepsilon}}_i(t_1 + \Delta t) \end{aligned} \tag{14.35}$$

Let the quantity $\Delta\tilde{\varepsilon}_i$ be determined as

$$\Delta\tilde{\varepsilon}_i = C_i \int_{t_1}^{t_1+\Delta t} e^{\frac{E_i}{\eta_i}\tau} \frac{d\sigma(\tau)}{d\tau} d\tau \tag{14.36}$$

then

$$\tilde{\varepsilon}_i(t_1 + \Delta t) = \tilde{\varepsilon}_i(t_1) + \Delta\tilde{\varepsilon}_i \tag{14.37}$$

Insertion of (14.37) in (14.35) and subtraction of (14.34) provide

$$\begin{aligned} \varepsilon(t_1 + \Delta t) &= \varepsilon(t_1) + (C_0 + \sum_{i=1}^{n} C_i)\Delta\sigma \\ &- \sum_{i=1}^{n}(e^{-\frac{E_i}{\eta_i}(t_1+\Delta t)} - e^{-\frac{E_i}{\eta_i}t_1})\tilde{\varepsilon}_i(t_1) - \sum_{i=1}^{n} e^{-\frac{E_i}{\eta_i}(t_1+\Delta t)}\Delta\tilde{\varepsilon}_i \end{aligned}$$

It appears that once $\varepsilon(t_1)$ and $\tilde{\varepsilon}_i(t_1)$ are known, all terms on the right-hand side are trivially identified except for the very last term that involves the quantity $\Delta\tilde{\varepsilon}_i$ which involves an integration. However, this integration is given by (14.36) and the important issue is that the integration limits are t_1 and $t_1 + \Delta t$, i.e the integration should only be performed over the current time step and not over the entire loading history.

The format (14.33) therefore allows a close prediction to experimental data and it implies a computational scheme that is very simple; for that reason this format is often adopted in the literature. In essence, it was suggested by Zienkiewicz and Watson (1966) and more information and generalizations to include aging material parameters are given by Bažant (1979, 1982, 1996) and Dahlblom (1987). Applications to orthotropic materials are discussed for wood by Mårtensson (1992) and Ormarsson (1999) and for paper by Lif (2003).

Let us as an example establish the matrices C_0 and C_i - or rather their tensorial counterparts - in (14.33) for isotropic materials. In analogy with (14.21) we take

$$\begin{aligned} C_{ijkl}^0 &= C_{ijkl} = \frac{1}{2G}[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{v}{1+v}\delta_{ij}\delta_{kl}] \\ C_{ijkl}^i &= \frac{1}{2G_i}[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\xi_i}{1+\xi_i}\delta_{ij}\delta_{kl}] \end{aligned}$$

For uniaxial stress conditions, where ε_{11} is the axial strain and $\varepsilon_{22} = \varepsilon_{33}$ is the transverse strain, (14.33) gives

$$\begin{aligned} \varepsilon_{11} &= (\frac{1}{2G}\frac{1}{1+v} + \sum_{i=1}^{n}\frac{1}{2G_i}\frac{1}{1+\xi_i})\sigma_{11} - \sum_{i=1}^{n} e^{-\frac{E_i}{\eta_i}t}\int_{-\infty}^{t} e^{\frac{E_i}{\eta_i}\tau}\frac{1}{2G_i}\frac{1}{1+\xi_i}\frac{d\sigma_{11}}{d\tau}d\tau \\ \varepsilon_{22} &= -(\frac{1}{2G}\frac{v}{1+v} + \sum_{i=1}^{n}\frac{1}{2G_i}\frac{\xi_i}{1+\xi_i})\sigma_{11} + \sum_{i=1}^{n} e^{-\frac{E_i}{\eta_i}t}\int_{-\infty}^{t} e^{\frac{E_i}{\eta_i}\tau}\frac{1}{2G_i}\frac{\xi_i}{1+\xi_i}\frac{d\sigma_{11}}{d\tau}d\tau \end{aligned}$$

It appears that if we choose $\xi_i = \nu$ then $\varepsilon_{22} = -\nu\varepsilon_{11}$ holds not only for elastic conditions, but also during development of creep strains; as previously discussed, this convenient situation is often supported experimentally, for instance for concrete, and it implies in (14.33) the great simplification that $\boldsymbol{C}_i = \frac{G}{G_i}\boldsymbol{C}_0$.

We finally observe that the format (14.33), in general, provides a very simple expression for the strain rate, namely

$$\dot{\boldsymbol{\varepsilon}}(t) = \boldsymbol{C}_0\dot{\boldsymbol{\sigma}}(t) + \sum_{i=1}^{n} \frac{E_i}{\eta_i} e^{-\frac{E_i}{\eta_i}t} \tilde{\boldsymbol{\varepsilon}}_i(t)$$

Let us finally observe that it is tradition in the literature on linear viscoelasticity to make use of the *Laplace-transform*, which is a convenient means to transform linear ordinary differential equations like (14.16) into algebraic equations and which also transforms convolution integrals like (14.28) to simple expressions in the Laplace-transforms. For convenience and in order to emphasize the physical aspects, we have here chosen not to make use of this approach.

The hereditary approach described above is derived from Boltzmann's superposition principle, i.e. it relies on linearity. However, it is possible to adopt a hereditary format and even develop a theory for nonlinear viscoelasticity. A number of possibilities exists and one approach is to use a multiple integral representation instead of the single integral appearing in (14.28). However, the formulations soon become very involved and applications for the solution of engineering problems seem to be scarce. Comprehensive reviews of various nonlinear hereditary theories are also included in the expositions of Findley *et al.* (1976) and Rabotnov (1980).