

# *Material Symmetry and Thermostatic Inequalities in Finite Elastic Deformations*

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## **Introduction**

This paper is concerned with *elastic materials*; these are substances for which the present stress  $S$  on a material point depends on only the present local configuration<sup>1</sup>  $M$  of that point:

$$S = S(M). \tag{1}$$

There are several highly developed branches of physics which rest on special cases of the constitutive assumption (1): Euler's hydrodynamics of perfect fluids, the classical theory of infinitesimal elastic deformations, and the modern theory of finite elastic deformations. Our present interest is not in the solution of special problems in these subjects but rather in the theory of the form of the function  $S$  itself; here we investigate the general limitations placed on  $S$  by material symmetry and thermodynamic considerations.

### *Applicability of the Theory*

Let us consider the situations in which we expect a theory based on (1) to be useful. It is known that there is a large, albeit not all-inclusive, class of

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<sup>1</sup> In Sections 1 and 2 we explain more precisely some of the mechanical and kinematical terms used in this Introduction with only abbreviated and heuristic descriptions.

substances, called *simple materials*, whose behavior in general mechanical processes is described by the assertion that the stress at a material point is determined by a function of the history of the local configuration of the point:

$$S = \underset{s=0}{\overset{\infty}{\mathfrak{S}}}(M^s). \quad (2)$$

Here  $M^s$  denotes the local configuration  $s$  seconds ago, and  $\mathfrak{S}$  is a functional whose argument is the function  $M^s$ ,  $0 \leq s < \infty$ , and whose value is the symmetric tensor  $S$ . Most of the literature on continuum mechanics deals with simple materials. For examples we have the theory of linear viscoelasticity, the hydrodynamics of perfect, viscous and non-Newtonian fluids, and also recent general theories of materials with nonlinear memory.<sup>2, 3</sup>

Now, when we consider a material point which has been in its present local configuration at all times in the past, the function  $M^s$  reduces to a constant,

$$M^s \equiv M = \text{const.}, \quad 0 \leq s < \infty, \quad (3)$$

and (2) yields the result that  $S$  depends only on  $M$ :

$$S = \mathbf{S}(M). \quad (4)$$

Thus, the theory of elastic materials describes simple materials which have always been at rest.

Suppose that the material described by the functional  $\mathfrak{S}$  in (2) has a "fading memory", *i.e.*, that  $\mathfrak{S}$  is continuous in such a way that  $S$  is "more sensitive" to changes in the local configuration  $M^s$  at small  $s$  (recent past) than to changes in the local configuration at large  $s$  (distant past).<sup>4</sup> Then, given any history  $M^s$ , we can construct *retarded histories*  $M^{\alpha s}$ ,  $0 < \alpha < 1$ , and prove that<sup>7</sup>

$$\lim_{\alpha \rightarrow 0} \underset{s=0}{\overset{\infty}{\mathfrak{S}}}(M^{\alpha s}) = \mathbf{S}(M^0). \quad (5)$$

The function  $\mathbf{S}$  in Eqs. (4) and (5) is the same function.

Equation (5) tells us that for every history  $M^s$  there exists a retarded history  $M^{\alpha s}$ , "essentially the same as  $M^s$  but slower", such that the present stress corresponding to  $M^{\alpha s}$  is given, to as good an approximation as we wish, by  $\mathbf{S}(M^0)$ . This result justifies, within the framework of the theory of simple materials, the use of the static stress-strain function  $\mathbf{S}$  in the discussion of slow processes; *i.e.*, it gives a motivation to the *dynamic* (as distinguished from the static) theories of perfect fluids and perfectly elastic solids.

#### *Thermostatic Inequalities*

In the theory of *infinitesimal* elastic deformations from a natural state (*i.e.* a state with zero stress), it is usually assumed that the stress-strain function  $\mathbf{S}$  is determined by the gradient of a *positive-definite* strain-energy function  $\sigma$ .

<sup>2</sup> GREEN, A. E., & R. S. RIVLIN, Arch. Rational Mech. Anal. 1, 1 (1957).

<sup>3</sup> NOLL, W., Arch. Rational Mech. Anal. 2, 197 (1958).

<sup>4</sup> This concept of "fading memory" is made precise in reference 5 and exploited further in reference 6.

<sup>5</sup> COLEMAN, B. D., & W. NOLL, Arch. Rational Mech. Anal. 6, 355 (1960).

<sup>6</sup> COLEMAN, B. D., & W. NOLL, Rev. Mod. Phys. 33, 239 (1961).

<sup>7</sup> The crucial step in proving (5) is Eq. (3.21) of reference 5.

It has often been supposed that the positive-definiteness of  $\sigma$  should be justifiable in terms of fundamental principles in classical thermodynamics, but rigorous arguments to this effect have not come forth. Nor has classical thermodynamics told us how to extend this assumption on  $\sigma$  (and hence on  $\mathbf{S}$ ) for infinitesimal elasticity to an assertion about  $\mathbf{S}$  that is in accord with experience in *finite* elasticity.<sup>8</sup>

Yet, Eq. (3) tells us that the function  $\mathbf{S}$  must be appropriate for those physical situations in which a simple material can be regarded as having been forever at rest, and  $\mathbf{S}$  must therefore describe "equilibrium states." Furthermore, our thermodynamic intuition suggests that an equilibrium state is one which maximizes an entropy or minimizes a free energy with respect to an appropriate class of disturbances, and this, in turn, suggests that the function  $\mathbf{S}$  should be subject to certain inequalities. The failure of classical thermodynamics to yield these inequalities is, in part, due to the difficulty in deciding what to include in the "class of disturbances".

In 1959<sup>9,10</sup> we proposed a theory of thermostatics which led to a general inequality on  $\mathbf{S}$ . We refer to that inequality as the TI (Thermostatic Inequality). In references 9 and 11 consequences of the TI were obtained for various types of materials. In the theory of infinitesimal deformations from a natural state, the TI reduces to the classical assumption of the positive-definiteness of  $\sigma$ . The consequences of the TI for finite elastic deformations seem to be in accord with all experience in solids.

For example, consider a homogeneous deformation of an undistorted isotropic elastic cube in the direction of its three axes. The TI then implies that the greater stretch occurs always in the direction of the greater applied force.<sup>9</sup>

For fluids, the TI is equivalent to the assertion that the pressure be positive and that the compression modulus be greater than two-thirds of the pressure. That the pressure is positive seems to be in accord with experience. However, for a fluid whose critical point occurs at a high pressure there can be a range of densities at which the compression modulus is less than two-thirds of the pressure.

Our TI is equivalent to a requirement of stability against homogeneous disturbances at fixed surface forces. Such a requirement should be appropriate for solids, because it is the surface forces that are controlled in most measurements on solids. For fluids, however, it is usually pressure, and not surface force, that is controlled. When the pressure is fixed during a deformation, the surface forces do not remain fixed but change their direction so as to stay normal to the surface and change their magnitude so as to compensate for the variation of the surface area. The familiar statement that the compression modulus is positive can be derived from a requirement of stability against variations in volume at fixed pressure. Such a requirement is suitable for fluids but does not yield adequate restrictions on the behavior of solids; it does not even yield the positivity of the shear modulus for isotropic infinitesimal elasticity.

<sup>8</sup> TRUESDELL, C., *Z. angew. Math. u. Mech.* **36**, 97 (1956).

<sup>9</sup> COLEMAN, B. D., & W. NOLL, *Arch. Rational Mech. Anal.* **4**, 97 (1959).

<sup>10</sup> A different formulation of the 1959 theory is given in reference 11.

<sup>11</sup> COLEMAN, B. D., *Arch. Rational Mech. Anal.* **9**, 172 (1962).

It is still an open question whether it is possible to find a single inequality from which one can derive all thermodynamic restrictions on the static behavior of elastic materials, whether they be fluids, solids, or neither.

In this article we make a detailed study of implications of the TI, stating, without proof, some of the known theorems and presenting several new theorems. Proofs are supplied for all new theorems stated.

Throughout the present discussion we ignore the dependence of the stress on a thermodynamic parameter, such as the temperature or the entropy density. Furthermore, we follow the procedure of a recent article by TRUESDELL & TOUPIN<sup>12</sup>, and, whenever possible, we work with a generalization of the TI, called the WTI (Weakened Thermostatic Inequality) which can be expressed directly in terms of the stress-strain function  $\mathbf{S}$  without recourse to mention of internal energy or free-energy functions. This deliberate suppression of mention of thermodynamic variables is done to emphasize the mechanical significance of our thermodynamic considerations.

Before going into the theory of the TI, we discuss the characterization of material symmetry in terms of stress-strain functions, giving simplified proofs of some known propositions and presenting several new results.

#### *On Notation*

The term *tensor* is used here as a synonym for linear transformation of a vector space into itself. We denote tensors by Latin majuscules:  $F, U, Q, R, I$ . We reserve the symbol  $X$ , however, to represent material points of a physical body. For the trace of a tensor  $F$  we write  $\text{tr } F$ , and for the determinant of  $F$  we write  $\det F$ . We say that  $F$  is invertible if  $F$  has an inverse  $F^{-1}$ , which is the case if and only if  $\det F \neq 0$ . The transpose of  $F$  is denoted by  $F^T$ ;  $F$  is symmetric if  $F^T = F$  and skew if  $F^T = -F$ . The "unit tensor" (*i.e.* the identity transformation) is denoted by  $I$ . If  $Q Q^T = Q^T Q = I$ , then  $Q$  is orthogonal. If  $Q$  is orthogonal and such that  $\det Q = +1$ , then  $Q$  is proper orthogonal. If the inner product  $\mathbf{v} \cdot U \mathbf{v}$  is strictly greater than zero for all nonzero vectors  $\mathbf{v}$ , then we say that the tensor  $U$  is positive-definite. If  $U$  is symmetric, then a necessary and sufficient condition that  $U$  be positive-definite is that all the proper numbers of  $U$  be positive.

### 1. Mechanical Preliminaries

For convenience, we briefly summarize some definitions and general results of continuum mechanics which are prerequisite to the present paper. Since detailed axiomatic treatments of this material have been given elsewhere,<sup>3,13,14</sup> the results stated here are asserted without proof.

<sup>12</sup> TRUESDELL, C., & R. A. TOUPIN, Arch. Rational Mech. Anal. **12**, 1 (1963). The inequality which we here call the "WTI" is called the "GCN condition" by TRUESDELL & TOUPIN.

<sup>13</sup> NOLL, W., in: Proceedings of the Berkeley Symposium on the Axiomatic Method, pp. 266–281. Amsterdam: North Holland.

<sup>14</sup> Another brief summary is given in §§ 1 and 5 of reference 9. For a thorough exposition of the foundations of continuum mechanics see C. TRUESDELL & R. A. TOUPIN, in: Encyclopedia of Physics, Edited by S. FLÜGGE, Vol. III/1. Berlin-Göttingen-Heidelberg: Springer 1959.

### Kinematics

A body  $\mathcal{B}$  is a smooth manifold whose elements  $X$  are called *material points*. A configuration  $\mathbf{f}$  of  $\mathcal{B}$  is a one-to-one mapping of  $\mathcal{B}$  onto a region  $\mathcal{R}$  in a three-dimensional Euclidean point space  $\mathcal{E}$ . The point  $\mathbf{x}=\mathbf{f}(X)$  is called the *position of  $X$*  in the configuration  $\mathbf{f}$ , and  $\mathcal{R}=\mathbf{f}(\mathcal{B})$  is called the *region* occupied by  $\mathcal{B}$  in  $\mathbf{f}$ . The *mass distribution*  $m$  of  $\mathcal{B}$  is a measure defined on subsets of  $\mathcal{B}$ . To each configuration  $\mathbf{f}$  of  $\mathcal{B}$  corresponds a mass density  $\rho$ .

If we have two configurations  $\mathbf{f}_1$  and  $\mathbf{f}_2$  of the same body  $\mathcal{B}$ , we can consider the composition  $\mathbf{g}_{2,1}$  of  $\mathbf{f}_2$  with the inverse  $\mathbf{f}_1^{-1}$  of  $\mathbf{f}_1$ ;  $\mathbf{g}_{2,1}$  is defined by

$$\mathbf{g}_{2,1}(\mathbf{x})=\mathbf{f}_2(\mathbf{f}_1^{-1}(\mathbf{x})).$$

Clearly,  $\mathbf{g}_{2,1}$  is a mapping of the region  $\mathcal{R}_1$  occupied by  $\mathcal{B}$  in the configuration  $\mathbf{f}_1$  onto the region  $\mathcal{R}_2$  occupied by  $\mathcal{B}$  in the configuration  $\mathbf{f}_2$ . The class of admissible configurations  $\mathbf{f}$  is assumed to be such that for any two configurations  $\mathbf{f}_1$  and  $\mathbf{f}_2$ ,  $\mathbf{g}_{2,1}$  is a smooth homeomorphism<sup>15</sup> of  $\mathcal{R}_1$  onto  $\mathcal{R}_2$ . Now, since  $\mathbf{g}_{2,1}$  is a mapping of one region of ordinary Euclidean space onto another, there is no difficulty in computing the gradient of  $\mathbf{g}_{2,1}$ ,  $\text{grad } \mathbf{g}_{2,1}$ . Because  $\mathbf{f}_1$  is one-to-one,  $\text{grad } \mathbf{g}_{2,1}$  can be regarded as either a field over  $\mathcal{R}_1$  or as a field over the manifold  $\mathcal{B}$ . Here it is convenient for us to take the latter point of view. We denote the values of  $\text{grad } \mathbf{g}_{2,1}$  by  $F_{2,1}(X)$ . Of course  $F_{2,1}(X)$  is a tensor; it is called the *deformation gradient* (at the material point  $X$ ) *of the configuration  $\mathbf{f}_2$  computed taking the configuration  $\mathbf{f}_1$  as reference*. Because the function  $\mathbf{g}_{2,1}$  is one-to-one, the tensors  $F_{2,1}(X)$  are invertible.

If we have three configurations  $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$  of the same body  $\mathcal{B}$  and compute the three deformation gradients  $F_{3,1}(X)$ ,  $F_{3,2}(X)$  and  $F_{3,1}(X)$ , then it follows from the chain rule for differentials of vector functions that

$$F_{3,1}(X)=F_{3,2}(X) F_{2,1}(X). \tag{1.1}$$

If  $F_{2,1}$  is independent of  $X$ , we say that  $\mathbf{f}_1$  and  $\mathbf{f}_2$  are related by a homogeneous deformation. If, in this case,  $F_{2,1}$  is a proper orthogonal tensor, then  $F_{2,1}$  is said to characterize a *rigid rotation*. If  $F_{2,1}$  is orthogonal (with no restriction on the sign of  $\det F_{2,1}$ ), then  $F_{2,1}$  is said to correspond to an *orthogonal transformation*, i.e. a combination of rigid rotations and reflections. If  $F_{2,1}$  has the form  $\alpha Q$ , where  $\alpha$  is a scalar and  $Q$  is orthogonal, then  $F_{2,1}$  describes a *similarity transformation*, i.e. a combination of an orthogonal transformation and a uniform expansion by the factor  $\alpha$ . If  $F_{2,1}$  is a positive definite symmetric tensor, then  $\mathbf{f}_2$  is said to be obtained from  $\mathbf{f}_1$  by a *pure stretch*; the proper vectors of  $F_{2,1}$  then give the *principal directions of stretch*, and the proper numbers of  $F_{2,1}$  are the *principal stretches*.

Let us return to deformations which are not necessarily homogeneous; such deformations can still be regarded, in a sense which we shall now make precise, as homogeneous on a "local level".

If, at a particular material point  $X$ ,  $F_{2,1}(X)$  is the unit tensor  $I$ , then we say that  $\mathbf{f}_1$  and  $\mathbf{f}_2$  give rise to the same local configuration at  $X$ . To state this

<sup>15</sup> A smooth homeomorphism is a one-to-one function  $\mathbf{g}$  such that both  $\mathbf{g}$  and its inverse  $\mathbf{g}^{-1}$  are continuously differentiable.

more formally, we observe that for each fixed  $X$  we can define an equivalence relation " $\sim$ " among all the configurations  $\mathbf{f}$  of  $\mathcal{B}$  by the condition that  $\mathbf{f}_1 \sim \mathbf{f}_2$  if  $F_{2,1}(X) = I$ ; the resulting equivalence classes of configurations are the *local configurations*  $M$  of  $X$ . If  $M_1$  is the equivalence class of  $\mathbf{f}_1$  at  $X$  and if  $M_2$  is the equivalence class of  $\mathbf{f}_2$  at  $X$ , then  $F_{2,1}(X)$  depends only on  $M_1$  and  $M_2$ ; it is also true that  $M_2$  is uniquely determined by  $F_{2,1}(X)$  and  $M_1$ . Hence we can call  $F_{2,1}(X)$  simply *the deformation gradient from  $M_1$  to  $M_2$* . Dropping the symbol  $X$  in  $F_{2,1}(X)$ , we write

$$M_2 = F_{2,1} M_1. \quad (1.2)$$

If we denote the mass densities at  $X$  corresponding to the local configurations  $M_1$  and  $M_2$  by  $\rho_1$  and  $\rho_2$ , respectively, then

$$\rho_2 = \frac{1}{|\det F_{2,1}|} \rho_1. \quad (1.3)$$

It is often convenient to employ a fixed *local reference configuration*  $M_0$  and to characterize all other local configurations

$$M = F M_0 \quad (1.4)$$

by their deformation gradients  $F$  from  $M_0$ . It follows from (1.4) that if

$$M_1 = F_1 M_0, \quad M_2 = F_2 M_0, \quad (1.5)$$

then the deformation gradient  $H$  from  $M_1$  to  $M_2$  is given by

$$H = F_2 F_1^{-1}, \quad F_2 = H F_1. \quad (1.6)$$

A theorem of algebra, called the *polar decomposition theorem*, states that any invertible tensor  $F$  can be written in two ways as the product of a symmetric positive-definite tensor and an orthogonal tensor:

$$F = R U, \quad (1.7)$$

$$F = V R. \quad (1.8)$$

Furthermore, the orthogonal tensor  $R$  and the symmetric positive-definite tensors  $U$  and  $V$  in these decompositions are uniquely determined by  $F$  and obey the following relations:

$$U^2 = F^T F, \quad (1.9)$$

$$V^2 = F F^T, \quad (1.10)$$

$$U = R^T V R. \quad (1.11)$$

Equations (1.7) and (1.8) have the following significance in kinematics: Any deformation gradient  $F$  may be regarded as being the result of a pure stretch  $U$  followed by an orthogonal transformation  $R$ , or an orthogonal transformation  $R$  followed by a pure stretch  $V$ . These interpretations uniquely determine the pairs  $R, U$  and  $R, V$ . The orthogonal transformations entering these two interpretations are the same; however, the pure stretches  $U$  and  $V$  can be different. It follows from (1.11) that although  $U$  and  $V$  may have different principal directions, they must have the same proper numbers  $u_i$ . We call the orthogonal

tensor  $R$  the *rotation tensor*<sup>16</sup> and the tensors  $U$  and  $V$ , respectively, the *right* and *left stretch tensors*. If  $M_2 = FM_1$ , then we call  $U$  and  $V$ , respectively, the *right* and *left stretch tensors from  $M_1$  to  $M_2$* , and the proper numbers  $u_i$  of  $U$  and  $V$  are called the *principal stretches from  $M_1$  to  $M_2$* .

### Stress

One distinguishes between contact forces and body forces. On assuming the principle of linear momentum and the principle of angular momentum, one can show, after laying down some strong continuity assumptions, that the contact force  $d\mathbf{c}$  across an oriented surface element with unit normal  $\mathbf{n}$  and area  $dA$  is given by

$$d\mathbf{c} = S\mathbf{n}dA, \quad (1.12)$$

where  $S$  is a symmetric tensor called the *stress tensor*.<sup>17</sup>

A proper vector of  $S$  is said to determine a *principal axis of stress*. Proper numbers of  $S$  are called *principal stresses*.

Since we are here interested in only "local phenomena", we ignore body forces throughout our discussion.

### Changes of Frame

A *change of frame* is the mathematical embodiment of the physical concept of a change of observer.

In this paper we assume that the stress tensor  $S$  at a material point  $X$  is determined by the present value of the local configuration  $M$  at that point. It follows that, for our present purposes, we can define a change of frame to be the following simultaneous transformation of the local configuration  $M_X$  and the stress  $S_X$  at each material point  $X$  of a body:

$$M_X \rightarrow QM_X, \quad (1.13a)$$

$$S_X \rightarrow QS_XQ^{-1}; \quad (1.13b)$$

here  $Q$  is an orthogonal tensor independent of  $X$ . The change of frame (1.13) is said to be characterized by  $Q$ .

In other words, a change of frame is a simultaneous orthogonal transformation of both the present configuration of  $\mathcal{B}$  and the present force system acting on  $\mathcal{B}$ .

The *principle of material objectivity* is a mathematization, for classical mechanics, of the physical idea that the behavior of a material should be independent of the observer. In our present theory this principle reduces to the statement that the dependence of the stress on the local configuration must be invariant under changes of frame; *i.e.*, this dependence must be such that if  $M$  is changed from  $M$  to  $QM$ , where  $Q$  is any orthogonal tensor, then the stress  $S$  corresponding to  $M$  must change to  $QSQ^{-1}$ .

## 2. The Response Function and the Isotropy Group

We are now prepared to state formally our

*Fundamental Constitutive Assumption.* To each local configuration  $M$  there corresponds a unique value of the stress tensor  $S$ .

<sup>16</sup> In applications  $R$  is usually proper orthogonal.

<sup>17</sup> For details, see reference 13.

Such a correspondence between  $S$  and  $M$  is called an *elastic material* (or, simply, "a material"). Since the behavior of a material should be independent of the observer, we also lay down an

*Assumption of Material Objectivity.* The dependence of the stress  $S$  on the local configuration  $M$  is such that if  $M_2 = QM_1$  where  $Q$  is any orthogonal tensor, then  $S_2 = QS_1Q^T$  where  $S_1$  and  $S_2$  are the stresses corresponding to  $M_1$  and  $M_2$ , respectively.

We follow standard procedure and express the dependence of  $S$  on  $M$  by picking a local reference configuration  $M_o$ , characterizing all local configurations by their deformation gradients  $F$  from  $M_o$  in accordance with (1.4), and then regarding  $S$  as a function  $g$  of  $F$ :

$$S = g(F). \quad (2.1)$$

The function  $g$  is called the *response function*, of our material, *taken relative to  $M_o$* .

It follows from our Assumption of Material Objectivity that  $g$  must obey the identity

$$g(QF) = Qg(F)Q^{-1} \quad (2.2)$$

for all tensors  $F$  and all orthogonal tensors  $Q$ . Using Eqs. (2.2) and (1.7), we infer that Eq. (2.1) is equivalent to

$$S = Rg(U)R^{-1}. \quad (2.3)$$

The function  $g$  depends on the choice of the reference configuration  $M_o$ . To exhibit the nature of this dependence of  $g$  on  $M_o$ , we note that it follows from (1.5) and (1.6) that if  $F$  and  $F'$  are two deformation gradients characterizing the same local configuration  $M$  with respect to two different local reference configurations  $M_o$  and  $M'_o$ , *i.e.*

$$M = FM_o = F'M'_o,$$

and if

$$M'_o = GM_o,$$

then

$$F = F'G.$$

If the response functions relative to  $M_o$  and  $M'_o$  are denoted by  $g$  and  $g'$ , respectively, we infer

$$S = g(F) = g'(F') = g'(FG^{-1}).$$

This proves

**Proposition 1.** *If  $g$  and  $g'$  are the response functions for the same material but taken relative to different local configurations  $M_o$  and  $M'_o$ , with  $M'_o = GM_o$ , then*

$$g(F) = g'(FG^{-1}) \quad (2.4)$$

for all  $F$ .

For a given material it may turn out that  $g$  remains the same function if the local reference configuration  $M_o$  is changed to another local reference configuration  $M'_o = HM_o$  with the same density, *i.e.*, that

$$g(F) = g'(F) \quad (2.5)$$



for all  $F$ . (The physical interpretation of the identity (2.5) is that the two configurations  $M_o$  and  $M'_o$  must be equal in their response to equal deformations, *i.e.*, that the classes of reference configurations for which the identity (2.5) holds must characterize the isotropy of the material under consideration.) Now, it follows from (2.4) that if (2.5) holds for  $M_o$  and  $M'_o = HM_o$ , then  $g$  satisfies the identity

$$g(F) = g(FH^{-1}) \quad (2.6)$$

for all  $F$ . Since we are here assuming that  $M'_o$  and  $M_o$  have the same density, it is clear that  $|\det H| = 1$ ; *i.e.*,  $H$  is here a unimodular transformation. If the identity (2.6) holds for  $H^{-1}$ , it holds also for  $H$ ; furthermore, the set of unimodular transformations  $H$  for which the identity (2.6) holds forms a group. Let us now state the following important formal definition whose physical motivation should, we feel, now be clear.

**Definition.** The group  $\mathcal{I}$  of unimodular tensors  $H$  for which the identity

$$g(FH) = g(F) \quad (2.7)$$

holds for all  $F$ , where  $g$  is the response function relative to the reference configuration  $M_o$ , is called the *isotropy group relative to  $M_o$* .

$\mathcal{I}$  depends on  $M_o$ , but we shall see that for a given material the isotropy groups  $\mathcal{I}$  and  $\mathcal{I}'$  relative to two distinct local reference configurations  $M_o$  and  $M'_o$  are conjugate and hence isomorphic. In fact, combination of (2.4) and (2.7) shows that

$$g'(FG^{-1}) = g'(FHG^{-1}) \quad (2.8)$$

for all  $F$ . If we put  $H' = GHG^{-1}$  and  $F' = FG^{-1}$ , then  $FHG^{-1} = F'H'$ , and (2.8) becomes the assertion that

$$g'(F') = g'(F'H') \quad (2.9)$$

for all  $F'$ , but this simply says that  $H'$  is in  $\mathcal{I}'$ . Hence we have

**Proposition 2.** *If  $G$  is the deformation gradient from  $M_o$  to  $M'_o$ , then  $H$  belongs to  $\mathcal{I}$  if and only if  $GHG^{-1}$  belongs to  $\mathcal{I}'$ , where  $\mathcal{I}$  and  $\mathcal{I}'$  are, respectively, the isotropy groups relative to  $M_o$  and  $M'_o$ .*

$\mathcal{I} = \mathcal{I}'$  would mean that the change of reference configuration  $G$  does not affect the isotropy group. By Proposition 2 this is the case if and only if  $G^{-1}HG$  belongs to  $\mathcal{I}$ . In the language of group theory, this condition states that  $G$  must belong to the *normalizer* group of  $\mathcal{I}$  within the full linear group  $\mathcal{L}$ . Thus we have

**Proposition 3.** *A change of reference configuration from  $M_o$  to  $GM_o$  leaves the isotropy group  $\mathcal{I}$  unchanged if and only if  $G$  belongs to normalizer of  $\mathcal{I}$  in  $\mathcal{L}$ .*

Suppose  $Q$  is orthogonal and belongs to  $\mathcal{I}$ ; then we can combine the identity (2.6), with  $H = Q$ , and the identity (2.2), to obtain

**Proposition 4.** *An orthogonal tensor  $Q$  belongs to the isotropy group  $\mathcal{I}$  if and only if*

$$Qg(F)Q^{-1} = g(QFQ^{-1}) \quad (2.10)$$

for all tensors  $F$ .

**Definitions.** If an elastic material has a local configuration  $M_0$  such that the corresponding isotropy group is either the orthogonal group  $\mathcal{O}$  or a group which contains  $\mathcal{O}$  as a subgroup, then we say that the material is *isotropic*, and  $M_0$  is called an *undistorted* local configuration. If a material is such that for some  $M_0$ ,  $\mathcal{I}$  is a subgroup (which need not be proper) of  $\mathcal{O}$ , then we say that the material is a *solid*, and we again call  $M_0$  *undistorted*. Hence for the undistorted configurations  $M_0$  of an *isotropic solid*  $\mathcal{I} = \mathcal{O}$ . In general, we say that a configuration  $M_0$  is *undistorted* if  $\mathcal{I}$  is comparable to  $\mathcal{O}$ ; i.e. if  $\mathcal{I}$  either is itself a subgroup of  $\mathcal{O}$  or contains  $\mathcal{O}$  as a subgroup. We say that a material is a *fluid* if  $\mathcal{I}$  is the full unimodular group  $\mathcal{U}$ . Since  $\mathcal{O}$  is a subgroup of  $\mathcal{U}$ , an elastic fluid is isotropic. It follows from Proposition 2 that if  $\mathcal{I} = \mathcal{U}$  for one local configuration  $M_0$ , then  $\mathcal{I}' = \mathcal{U}$  for every other local configuration  $M'_0$ ; hence, every configuration of a fluid is undistorted.

*Note.* Continuum mechanics does not yet have a standard terminology for the various mathematical concepts behind the intuitive notion of "isotropy". The particular definitions we give here are specializations to elastic materials of the definitions given by NOLL<sup>3</sup> for general simple materials. The reader is cautioned that, whereas we use *isotropic* as a quality of a *material*, other writers regard *isotropic* as a quality of both a *material and a configuration*. When these writers say that "the material is isotropic in the configuration  $M$ ", we say that " $M$  is an undistorted configuration of the isotropic material".

The following proposition is an immediate consequence of Proposition 4.

**Proposition 5.** *The response function  $g$  of an isotropic material, when taken relative to an undistorted state, obeys the following identity for all tensors  $F$  and all orthogonal tensors  $Q$ :*

$$g(QFQ^{-1}) = Qg(F)Q^{-1}. \quad (2.11)$$

Tensor-valued functions obeying (2.11) are called *isotropic functions*.

The next three propositions illustrate some of the physical motivation behind our formal definitions.

**Proposition 6.** *The stress on an undistorted state of an isotropic material is always a hydrostatic pressure.*

This proposition states that if  $\mathcal{I}$  contains  $\mathcal{O}$  as a subgroup (properly or improperly), then  $g(I) = -pI$ .

**Proof.** By hypothesis and Proposition 5 we have

$$Qg(I)Q^{-1} = g(QIQ^{-1}) = g(I)$$

for all orthogonal tensors  $Q$ ; i.e.,  $g(I)$  must commute with all orthogonal tensors. This is possible only if  $g(I)$  is a scalar multiple ( $-pI$ ) of the unit tensor  $I$ , *q.e.d.*

**Proposition 7.** *The stress on an elastic fluid is always a hydrostatic pressure  $-pI$  which depends on only the density  $\varrho$ .*

*In other words, for a simple fluid,*

$$g(F) = -p(\varrho)I \quad (2.12)$$

where

$$\varrho = \frac{1}{|\det F|} \varrho_0. \quad (2.13)$$

$\varrho$  being the density of the present local configuration  $FM$  and  $\varrho_0$  the density of the local reference configuration  $M_0$ .

**Proof.** For any given tensor  $F$ , the tensor  $H = |\det F| F^{-1}$  is obviously unimodular and hence belongs to the isotropy group  $\mathcal{S} = \mathcal{U}$  of the fluid. Therefore, by (2.7),

$$g(F) = g(FH) = g(|\det F| I)$$

depends only on  $|\det F|$  or, equivalently, on the density  $\varrho$ . That  $g(F)$  is a hydrostatic pressure for all  $M$  and  $F$  is a consequence of Proposition 6 and the fact that a simple fluid is an isotropic material all of whose configurations are undistorted.

**Proposition 8.** *Two undistorted local configurations of an isotropic solid can differ by only a similarity transformation.*

**Proof.** By definition, the isotropy group of an isotropic solid, relative to an undistorted reference configuration, is the orthogonal group  $\mathcal{O}$ . Hence, by Proposition 3, a change  $G$  from one undistorted reference configuration to another must belong to the normalizer of  $\mathcal{O}$  in  $\mathcal{L}$ . But it is a known result in group theory that the normalizer of  $\mathcal{O}$  is the group of all similarity transformations,<sup>18</sup> *q.e.d.*

### 3. Undistorted States of Aeolotropic Solids

Proposition 6 of Section 2 shows that the stress on an undistorted local configuration of an isotropic material (solid or not) must be a hydrostatic pressure. Here we investigate the limitations material symmetry imposes upon the stress on undistorted configurations of various types of aeolotropic solids. We also extend Proposition 8 of Section 2 to obtain limitations on the possible strains which can relate undistorted states of general solids.

As far as purely mechanical behavior is concerned, the symmetry of an elastic solid is characterized by a corresponding "type" of isotropy group  $\mathcal{S}$ . By our definition of the term *solid*,  $\mathcal{S}$  will be a subgroup of the orthogonal group  $\mathcal{O}$  if an undistorted local configuration  $M_0$  is used as a reference. If  $M'_0 = GM_0$  is another undistorted reference configuration, then the isotropy group  $\mathcal{S}'$  corresponding to  $M'_0$  will also be a subgroup of the orthogonal group. Moreover, according to Proposition 3 of Section 2,  $\mathcal{S}'$  will be the conjugate  $\mathcal{S}' = G\mathcal{S}G^{-1}$  of  $\mathcal{S}$  under  $G$ ; *i.e.*, every transformation  $Q'$  in  $\mathcal{S}'$  will be of the form

$$Q' = GQG^{-1} \quad (3.1)$$

where  $Q$  is a member of  $\mathcal{S}$ . Let

$$G = RU \quad (3.2)$$

be the polar decomposition of  $G$ , so that  $U$  is the right stretch tensor and  $R$  the rotation tensor of the deformation carrying  $M_0$  into  $M'_0$ . Combining Eqs. (3.1) and (3.2), we find

$$\begin{aligned} Q'G &= GQ, \\ Q'RU &= RUQ = (RQ)(Q^{-1}UQ). \end{aligned} \quad (3.3)$$

<sup>18</sup> Cf. H. WEYL, *The Classical Groups*, p. 22. Princeton 1946.

Since both  $Q'R$  and  $RQ$  are orthogonal tensors while  $U$  and  $Q^{-1}UQ$  are both positive-definite and symmetric, (3.3) gives two polar decompositions of the one tensor  $Q'G$ . Therefore, the uniqueness of a polar decomposition implies that

$$Q'R = RQ \quad \text{and} \quad U = Q^{-1}UQ. \quad (3.4)$$

The first of the Eqs. (3.4) shows that every member  $Q'$  of  $\mathcal{S}'$  is of the form

$$Q' = RQR^{-1}, \quad (3.5)$$

*i.e.*, that  $\mathcal{S}'$  is actually a conjugate  $\mathcal{S}' = R\mathcal{S}R^{-1}$  of  $\mathcal{S}$  within the orthogonal group. We say that two subgroups of the orthogonal group are of the same type if they are conjugate in this sense. The isotropy group of a solid depends on the choice of reference configuration, even when this configuration is restricted to be undistorted. The type to which this group belongs, however, represents an intrinsic property of the material.

Crystalline solids are classified into thirty-two classes, each of which is characterized by a certain type of symmetry group. The symmetry groups are finite subgroups of the orthogonal group. On putting  $Q = -I$  in (2.2) we see that

$$g(F(-I)) = (-I)g(F)(-I) = g(F), \quad (3.6)$$

which shows that the inversion  $-I$  is always a member of the isotropy group. This inversion, however, does not belong to the symmetry group of some of the crystal classes.<sup>19</sup> The following assumption seems to be natural:

*Hypothesis on Crystalline Solids.* Consider a crystal which, relative to some undistorted state  $M_o$ , has the crystallographic symmetry group (*i.e.* point group)  $\mathcal{C}$ . The isotropy group  $\mathcal{S}$  of this crystal, relative to  $M_o$ , is assumed to be the group generated by  $\mathcal{C}$  and the inversion  $-I$ .

The 32 types of symmetry groups<sup>20</sup> give rise to only 11 types of isotropy groups. In describing a group  $\mathcal{S}$  we need not list all the elements of  $\mathcal{S}$  but only a set of *generators* of  $\mathcal{S}$ , *i.e.* a set of members of  $\mathcal{S}$  which, when they and their inverses are multiplied among themselves in various combinations, yield all the elements of  $\mathcal{S}$ . For an isotropy group relative to an undistorted state of a solid it is always possible to find a list of generators  $-I, Q_1, \dots, Q_m$  such that each  $Q_i$  is *proper orthogonal*. We denote by  $R_n^\varphi$  the right-handed rotation through the angle  $\varphi$ ,  $0 < \varphi < 2\pi$ , about an axis in the direction of the unit vector  $\mathbf{n}$ . Each proper orthogonal transformation  $\neq I$  is a rotation of the form  $R_n^\varphi$ . Table 1 gives a description of the 11 types of isotropy groups for crystals. In this table,  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  denotes a right handed orthonormal basis and  $\mathbf{d} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$ .

A change from one group to another of the same type corresponds merely to a change of the orthonormal basis  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ .

<sup>19</sup> If we were dealing here with vector phenomena, as in electromagnetic theory, then we should not automatically have  $-I$  in  $\mathcal{S}$ , and we should have instead  $\mathcal{S} = \mathcal{C}$ . In other words, it is because our present theory involves only tensors of order two that our isotropy groups are sometimes bigger than the crystallographic point groups.

<sup>20</sup> Complete lists of members for all these groups are given by G. F. SMITH & R. S. RIVLIN, *Trans. Am. Math. Soc.* **88**, 175 (1958). The names we employ for the various crystal systems and classes are those used by SMITH & RIVLIN, who, in turn, state that they come from the 1952 Edition of DANA's *Textbook of Mineralogy*, revised by C. S. HURLBUT (New York: John Wiley).

Table 1. *Isotropy Groups for the Various Crystal Classes*

Ref. No.	Crystal Class	Proper Orthogonal Generators of $\mathcal{I}$	Number of Elements in $\mathcal{I}$
	<i>Triclinic System</i>		
1.	all classes	$I$	2
	<i>Monoclinic System</i>		
2.	all classes	$R_{\mathbf{k}}^{\pi}$	4
	<i>Rhombic System</i>		
3.	all classes	$R_{\mathbf{i}}^{\pi}, R_{\mathbf{j}}^{\pi}$	8
	<i>Tetragonal System</i>		
4.	{ tetragonal-disphenoidal tetragonal-pyramidal tetragonal-dipyramidal }	$R_{\mathbf{k}}^{2\pi/3}$	8
5.	{ tetragonal-scalenohedral ditetragonal-pyramidal tetragonal-trapezohedral ditetragonal-dipyramidal }	$R_{\mathbf{k}}^{\pi/2}, R_{\mathbf{i}}^{\pi}$	16
	<i>Cubic System</i>		
6.	{ tetratoidal diploidal }	$R_{\mathbf{i}}^{\pi}, R_{\mathbf{j}}^{\pi}, R_{\mathbf{d}}^{2\pi/3}$	24
7.	{ hextetrahedral gyroidal hexoctahedral }	$R_{\mathbf{i}}^{\pi/2}, R_{\mathbf{j}}^{\pi/2}$	48
	<i>Hexagonal System</i>		
8.	{ trigonal-pyramidal rhombohedral }	$R_{\mathbf{k}}^{2\pi/3}$	6
9.	{ ditrigonal-pyramidal trigonal-trapezohedral hexagonal-scalenohedral }	$R_{\mathbf{k}}^{2\pi/3}, R_{\mathbf{i}}^{\pi}$	12
10.	{ trigonal-dipyramidal hexagonal-pyramidal hexagonal-dipyramidal }	$R_{\mathbf{k}}^{\pi/3}$	12
11.	{ ditrigonal-dipyramidal dihexagonal-pyramidal hexagonal-trapezohedral dihexagonal-dipyramidal }	$R_{\mathbf{k}}^{\pi/3}, R_{\mathbf{i}}^{\pi}$	24

With respect to their elastic behavior alone, crystals can show only the 11 types of symmetry characterized by the 11 types of isotropy groups.

A type of symmetry appropriate not to crystals but to materials with a bundled or laminated structure or to drawn fibers is *transverse isotropy*. It is defined by the assumption that the isotropy group consists of  $\pm I$  and  $\pm R_{\mathbf{k}}^{\varphi}$ , for some fixed unit vector  $\mathbf{k}$ , and all angles  $\varphi$ ,  $0 < \varphi < 2\pi$ .

Consider an elastic solid and a particular undistorted reference configuration  $M_0$ . Denote the corresponding response function by  $\mathfrak{g}$  and the corresponding isotropy group by  $\mathcal{I}$ . The stress on  $M_0$  is given by

$$S_0 = \mathfrak{g}(I). \quad (3.7)$$

The following is an immediate consequence of Eq. (2.10) for  $F=I$ :

**Proposition 1.** *The stress  $S_o$  on an undistorted local configuration  $M_o$  of a solid commutes with every member of the isotropy group  $\mathcal{I}$  for  $M_o$ ; i.e.*

$$QS_o = S_oQ \quad (3.8)$$

holds for all  $Q$  in  $\mathcal{I}$ .

It is easily seen that (3.8) holds for all  $Q$  in  $\mathcal{I}$  if it holds for all rotations  $Q_i$ ,  $i=1, \dots, m$  in a list  $-I, Q_1, \dots, Q_m$  of generators of  $\mathcal{I}$ . Therefore, the problem of finding the possible stresses  $S_o$  on an undistorted state reduces to the problem of finding those symmetric tensors that commute with a certain set of rotations of the form  $R_n^\varphi$ . For each of the 11 cases given in Table 1 and for the case of transverse isotropy, this problem can easily be solved with the help of the following

**Commutation Theorem<sup>21</sup>.** *The symmetric tensor  $S$  commutes with the orthogonal tensor  $Q$  if and only if  $Q$  leaves each of the characteristic spaces of  $S$  invariant, i.e., if it maps each characteristic space into itself.*

Here, a *characteristic space* of  $S$  is defined to be a maximal subspace (of the ordinary three-dimensional vector space) consisting of only proper vectors of  $S$ . If  $S$  has three distinct proper numbers, then it has three one-dimensional characteristic spaces. If  $S$  has only two distinct proper numbers, then it has two characteristic spaces, one of which is one-dimensional and the other two-dimensional. In this case,  $S$  must be of the form<sup>22</sup>

$$S = -pI + qn \otimes n, \quad (3.9)$$

where  $-p$  and  $q-p$  are the proper numbers of  $S$  and  $n$  is a unit vector which generates the one-dimensional characteristic space. If the three proper numbers of  $S$  coincide, then the entire three-dimensional vector space is the only characteristic space of  $S$ , and  $S$  is of the form

$$S = -pI. \quad (3.10)$$

Now if  $\varphi \neq \pi$ , the only spaces left invariant by the rotation  $R_n^\varphi$  are the one-dimensional space of all multiples of  $n$  the two-dimensional space of all vectors perpendicular to  $n$ , and the entire three-dimensional vector space. The rotation  $R_n^\pi$  leaves invariant, in addition, each one-dimensional space generated by a vector perpendicular to  $n$ .

Proposition 1, Table 1, and the results from geometrical linear algebra just described enable one to establish easily the results collected in Table 2. In this table,  $i, j, k$  is the same orthonormal basis as is used for the isotropy groups in Table 1. Recall that a proper vector of the stress  $S_o$  determines a principal axis of stress.

Let us now return to Eqs. (3.4). It follows from the second of these equations that  $QU = UQ$ , which is the content of

<sup>21</sup> This theorem is a corollary to Theorem 2, p. 77, and Theorem 3, p. 157, of P. R. HALMOS, *Finite-Dimensional Vector Spaces*. Princeton: Van Nostrand, 2nd ed., 1958.

<sup>22</sup> The symbol  $\otimes$  denotes a tensor product; i.e.,  $n \otimes n$  is the tensor with the property that  $(n \otimes n)v = n(n \cdot v)$  for all vectors  $v$ .

**Proposition 2.** *Let  $M_0$  and  $M'_0$  be two local reference configurations of a solid, and let  $\mathcal{I}$  be the isotropy group relative to  $M_0$ . If  $M_0$  is undistorted, then  $M'_0$  is undistorted if and only if the right stretch tensor  $U$  from  $M_0$  to  $M'_0$  commutes with every member of  $\mathcal{I}$ .*

Table 2. Restrictions on the Stress for Undistorted States of Aeolotropic Solids

Type of Isotropy	Ref. No. in Table 1	Restriction on $S_0$
Triclinic system	1	no restriction
Monoclinic system	2	$\mathbf{k}$ is a proper vector of $S_0$
Rhombic system	3	$\mathbf{i}, \mathbf{j}, \mathbf{k}$ are proper vectors of $S_0$
{ Tetragonal system Hexagonal system Transverse isotropy	{ 4, 5, 8, 9, 10, 11 }	$S_0 = -pI + q\mathbf{k} \otimes \mathbf{k}$
Cubic system	6, 7	$S_0 = -pI$

It follows from this proposition that if  $M_0$  is an undistorted configuration of a solid with the symmetry listed in the first column of Table 2, then a necessary and sufficient condition that  $M'_0$  be an undistorted configuration of that solid is that the right stretch tensor  $U$  relating  $M'_0$  to  $M_0$  obey the restrictions listed for  $S_0$  in the second column.

#### 4. Thermostatic Inequalities

As we mentioned in the Introduction, there are reasons, whose origins lie outside of mechanics, for believing that stress-strain functions used in physical applications should be subjected to restrictions beyond those following from Material Objectivity. These restrictions should have the form of inequalities and, we believe, should follow from a properly formulated theory of the thermodynamics of continuous media.

The inequalities we wish to consider here can be most simply expressed through use of the first Piola-Kirchhoff stress tensor,  $T$ , defined by

$$T = |\det F| (F^{-1}S)^T = \frac{\rho_0}{\rho} SF^{T^{-1}}, \quad (4.1)$$

where  $S$  is the ordinary stress and  $F$  the deformation gradient. The definition of  $T$  depends on the choice of the reference configuration.  $T$  need not be symmetric. We denote the relation between  $T$  and  $F$  by

$$T = \mathfrak{h}(F), \quad (4.2)$$

where the response function  $\mathfrak{h}$  is related to the response function  $g$  of Eq. (2.1) by

$$\mathfrak{h}(F) = |\det F| g(F)F^{T^{-1}}. \quad (4.3)$$

When the equation of material objectivity (2.2) is expressed in terms of  $\mathfrak{h}$ , it reads

$$\mathfrak{h}(QF) = Q\mathfrak{h}(F). \quad (4.4)$$

Using Eqs. (4.4) and (1.7), we see that Eq. (4.2) is equivalent to

$$T = R \mathfrak{h}(U), \quad (4.5)$$

where  $R$  is the rotation tensor and  $U$  the right stretch tensor corresponding to  $F$ .

In a work on the foundations of thermostatics<sup>9</sup> we have laid down postulates which, in our present theory, are equivalent to the following

*Thermostatic Inequality (TI).* There exists a (scalar-valued) *energy function*  $\sigma$  such that

$$\sigma(F^*) - \sigma(F) - \text{tr}\{(F^* - F) \mathfrak{h}(F)^T\} > 0 \quad (4.6)$$

for all pairs of deformation gradients  $F, F^*$  such that  $F^* \neq F$  and  $F^* F^{-1}$  is positive-definite and symmetric.

The energy function  $\sigma(F)$  in (4.6) can be interpreted as either the internal energy density (per unit volume in the reference configuration) or the Helmholtz free energy density, depending on whether one regards the entropy density or the temperature as the thermodynamic quantity being held fixed, and then suppressing, in defining  $\mathfrak{g}$ .

In reference 9 it is shown that (4.6) implies

$$\mathfrak{h}(F)^T = \text{grad}_F \sigma(F). \quad (4.7)$$

Once (4.7) is established, our assumption of Material Objectivity, *i.e.* (2.2), is equivalent to the assertion that  $\sigma$  in (4.6) obey the identity

$$\sigma(QF) = \sigma(F) \quad (4.8)$$

for all  $F$  and all orthogonal  $Q$ .<sup>23</sup>

An alternative formulation of our TI is the following assertion.<sup>24</sup> Consider the Class  $\mathfrak{C}$  of continuous rectifiable curves  $F_t$  in the space of all invertible tensors  $F$ , and let the parameter  $t$  for these curves vary from 0 to 1. Let  $\mathfrak{C}'$  be the set of all curves  $F_t$  in  $\mathfrak{C}$  for which  $F_1 \neq F_0$  and  $F_1 F_0^{-1}$  is both symmetric and positive-definite. Then, the following inequality must hold for curves  $F_t$  in  $\mathfrak{C}'$ :

$$\int_{F_0}^{F_1} \text{tr}\{\mathfrak{h}(F_t)^T dF_t\} > \text{tr}\{(F_1 - F_0) \mathfrak{h}(F_0)^T\}. \quad (4.9)$$

The integral on the left in (4.9) is to be interpreted as a line integral along the curve  $F_t$  from  $t=0$  to  $t=1$ .

The quantities appearing on each side of (4.9) represent work, per unit volume in the reference configuration, done against contact forces at a material point  $X$  as the local configuration of  $X$  is deformed along path  $F_t M_0$ ,  $0 \leq t \leq 1$ . The quantity on the left in (4.9) gives the "true" work done, *i.e.* the work done assuming that at each  $t$  the contact forces on each material surface at  $X$  are those which one calculates using the stress tensor  $S = \mathfrak{g}(F_t)$  and the actual

<sup>23</sup> In essence, the proof of (4.8) is given in Theorem I of W. NOLL, J. Rational Mech. Anal. 4, 3 (1955); see also § 1 of reference 11.

<sup>24</sup> The proof that (4.9) is equivalent to the assertion that  $\sigma$  exists and obeys (4.5) is given in reference 11. In writing (4.9) we have made use of the fact that  $\mathfrak{g}(F)$  is symmetric. This enables us to simplify Eqs. (1.4) and the inequality (2.3) of reference 11.



configuration of the surface at time  $t$ ; the quantity on the right in (4.9), however, gives the work which would be done along the path  $F_t M_o$  if the contact forces were to remain fixed at their initial values.<sup>25</sup> Thus, (4.9) states that  $\mathfrak{h}$  must be such that contact forces always change in a process which results in a pure stretch, and, furthermore, they always change in such a way that the work done against them is greater than that which would have been done had they remained fixed.

Let us now return to (4.6). Of course, when  $F^*F^{-1}$  is symmetric and positive definite, so is  $FF^{*-1}$ , and we can interchange  $F$  and  $F^*$  in (4.6). If we do this and add the resulting inequality to (4.6), the terms involving  $\sigma$  cancel, and we get, as do TRUESDELL & TOUPIN<sup>12</sup>,

$$\text{tr}\{(F^* - F)[\mathfrak{h}(F^*) - \mathfrak{h}(F)]^T\} > 0. \quad (4.10)$$

We now state the

*Weakened Thermostatic Inequality* WTI. The response function  $\mathfrak{h}$  is such that, for all pairs of tensors  $F^*, F$  for which  $F^* \neq F$  and  $F^*F^{-1}$  is positive-definite and symmetric, we have the inequality (4.10).

The WTI is equivalent to demanding that

$$\text{tr}\{(U - I)F[\mathfrak{h}(UF) - \mathfrak{h}(F)]^T\} > 0 \quad (4.11)$$

for all  $F$  and all positive-definite symmetric  $U \neq I$ . Since the reference configuration can always be chosen such that  $F = I$ , the WTI is also equivalent to the requirement that

$$\text{tr}\{(U - I)[\mathfrak{h}(U) - \mathfrak{h}(I)]\} > 0 \quad (4.12)$$

for all positive-definite symmetric  $U$ , no matter what reference configuration is used in defining the response function  $\mathfrak{h}$ .

Of course the TI implies the WTI; the WTI does not imply TI. It is possible to find (theoretical) materials obeying our WTI but yet such that  $\mathfrak{h}$  is not derivable from an energy function  $\sigma$  through (4.7).

*Remark.* To say that a material obeys the TI is equivalent to the assertion that the response function  $\mathfrak{h}$  is such that (4.9) holds for *all* curves  $F_t$  in  $\mathfrak{C}$ . To say that a material obeys the WTI, however, is to assert only that  $\mathfrak{h}$  is such that (4.9) holds for those curves  $F_t$  in  $\mathfrak{C}'$  which are straight lines.<sup>26</sup>

It follows from these observations that any special result derived from the WTI inequality (4.10) will also hold for materials obeying the TI inequality (4.6). Since most of the known implications of the TI, and also the implications of the TI which we wish to report here, can hold also under the weaker (*i.e.* more general) assumption of the WTI, we shall base our present discussion of thermostatics upon the WTI. Throughout the rest of this paper, if the WTI is not mentioned in the statement of a proposition, it is to be understood to be present as an axiom.

<sup>25</sup> Note that, when the configuration of a surface is changing, keeping contact forces fixed is not equivalent to keeping the stress tensor  $S$  fixed; rather, it is equivalent to keeping the Piola-Kirchhoff tensor  $T$  fixed.

<sup>26</sup> BRAGG, L. E., & B. D. COLEMAN, J. Math. Phys. 4, 1074 (1963).

### 5. States of Pressure

Here we obtain some results which show that the WTI places limitations on the properties of local configurations which give rise to a hydrostatic pressure. Our main new result is Theorem 3, which tells us, among other things, that the stress tensor can vanish only in an undistorted configuration of a solid.

**Lemma.** *Consider two local configurations  $M_1$  and  $M_2$  of the same material, and suppose that they both correspond to hydrostatic pressures:*

$$S_1 = -p_1 I, \quad (5.1a)$$

$$S_2 = -p_2 I. \quad (5.1b)$$

Here  $p_1$  may or may not equal  $p_2$ . Consider the right stretch  $U$  from  $M_1$  to  $M_2$ ; i.e., put

$$M_2 = F M_1, \quad F = R U, \quad (5.2)$$

where  $R$  is orthogonal and  $U$  positive-definite and symmetric. Then, the following inequality must hold whenever  $U \neq I$ :

$$p_1(\operatorname{tr} U - 3) + p_2(\det U)(\operatorname{tr} U^{-1} - 3) > 0. \quad (5.3)$$

**Proof.** We use  $M_1 = M_0$  as the reference configuration. By Eqs. (4.1), the Piola-Kirchhoff tensors corresponding to  $M_1$  and  $M_2$  are

$$T_1 = S_1 = -p_1 I, \quad (5.4a)$$

$$T_2 = |\det F| (F^{-1} S_2)^T = -p_2(\det U) R U^{-1}. \quad (5.4b)$$

Hence, by (4.5), we obtain

$$\mathfrak{h}(I) = -p_1 I, \quad \mathfrak{h}(U) = -p_2(\det U) U^{-1}. \quad (5.5)$$

Substitution of Eqs. (5.5) into the form (4.12) of the WTI gives

$$\operatorname{tr}\{(U - I)[(-p_2)(\det U)U^{-1} + p_1 I]\} > 0, \quad (5.6)$$

whenever  $U \neq I$ ; (5.6) is equivalent to (5.3), *q.e.d.*

**Theorem 1.** *Suppose that two local configurations  $M_1$  and  $M_2$  of a material correspond to the same hydrostatic stress  $-pI$  with  $p > 0$ . Suppose further that  $M_1$  and  $M_2$  differ by more than an orthogonal transformation, i.e., that the principal stretches  $u_i$ ,  $i = 1, 2, 3$  from  $M_1$  to  $M_2$  are not all 1. Then the following two situations are both impossible:*

$$u_i \geq 1 \quad \text{for all } i, \quad (5.7a)$$

$$u_i \leq 1 \quad \text{for all } i. \quad (5.7b)$$

**Proof.** We use the Lemma, putting  $p_1 = p_2 = p$  in (5.1). Of course, the  $u_i$  mentioned here are just the proper numbers of  $U$  defined in (5.2). Now the inequality (5.3) states that if  $U \neq I$ , i.e. if the  $u_i$  are not all 1, then

$$p g > 0, \quad (5.8)$$

where

$$\begin{aligned} g &= (\operatorname{tr} U - 3) + (\det U)(\operatorname{tr} U^{-1} - 3) \\ &= (u_1 + u_2 + u_3 - 3) + (u_2 u_3 + u_1 u_3 + u_1 u_2 - 3 u_1 u_2 u_3). \end{aligned} \quad (5.9)$$

Since, by hypothesis,  $p > 0$ , the inequality (5.8) implies

$$g > 0. \quad (5.10)$$

Differentiation of Eq. (5.9) yields equations of the type

$$\frac{\partial g}{\partial u_j} = u_k + u_l + 1 - 3u_k u_l, \quad (5.11)$$

where  $j, k$ , and  $l$  are all distinct. Suppose now that the inequalities (5.7a) hold. We then also have inequalities of the form

$$u_k u_l \geq \frac{u_k + u_l + 1}{3}, \quad k \neq l, \quad (5.12)$$

and hence, by Eq. (5.11),

$$\frac{\partial g}{\partial u_j} \leq 0, \quad j = 1, 2, 3. \quad (5.13)$$

Since  $g$  reduces to 0 when  $u_1 = u_2 = u_3 = 1$ , it follows from (5.13) that  $g \leq 0$  when (5.7a) holds, which contradicts (5.10). Hence the inequalities (5.7a) are impossible. To show that the inequalities (5.7b) are also impossible, we need only interchange the roles of the configurations  $M_1$  and  $M_2$ , *q.e.d.*

**Corollary.** *Suppose two local configurations  $M_1$  and  $M_2$  give rise to the same positive hydrostatic pressure  $p$  and differ only by a similarity transformation  $F = \alpha Q$ ,  $Q$  orthogonal,  $\alpha > 0$ . Then  $\alpha = 1$ ; i.e.,  $M_1$  and  $M_2$  must have the same density and can differ by only a rotation or reflection.*

**Theorem 2.** *Suppose that, for a given material,  $M_1$  and  $M_2$  are two local configurations which have the same density and give rise to hydrostatic pressures  $p_1$  and  $p_2$ , respectively, so that Eqs. (5.1) hold. If  $p_1 \leq 0$  and  $p_2 \leq 0$ , then  $M_1$  and  $M_2$  can differ by only an orthogonal transformation, and we must have  $p_1 = p_2$ .*

**Proof.** Since here  $M_1$  and  $M_2$  have the same density, the right stretch tensor  $U$  of (5.2) satisfies

$$\det U = \det U^{-1} = 1. \quad (5.14)$$

Thus, if  $U \neq I$ , the inequality (5.3) becomes

$$p_1(\operatorname{tr} U - 3) + p_2(\operatorname{tr} U^{-1} - 3) > 0. \quad (5.15)$$

The arithmetic mean of a set of positive numbers is strictly greater than the geometric mean unless the numbers are all equal to one. Applied to the proper numbers of a positive-definite, symmetric tensor  $A$ , this observation states that

$$\sqrt[3]{\det A} < \frac{\operatorname{tr} A}{3} \quad \text{if } A \neq I. \quad (5.16)$$

On putting  $A = U$  and  $A = U^{-1}$  in (5.16) and using (5.14), we obtain

$$\operatorname{tr} U - 3 > 0, \quad \operatorname{tr} U^{-1} - 3 > 0, \quad (5.17)$$

whenever  $U \neq I$ . If  $p_1 \leq 0$  and  $p_2 \leq 0$ , the inequalities (5.15) and (5.17) are inconsistent. Hence  $U = I$ , which means that  $M_1$  and  $M_2$  can differ only by an orthogonal transformation  $F = R$  (see Eq. (5.2)). Moreover, using Eqs. (2.1)

and (2.2) with  $M_1$  as the reference configuration,  $F=I$  and  $Q=R$ , we find

$$S_1 = -p_1 I = g(I), \quad (5.18a)$$

$$S_2 = -p_2 I = g(R) = Rg(I)R^{-1} = R(-p_1 I)R^{-1} = -p_1 I; \quad (5.18b)$$

i.e.  $p_1 = p_2$ , q.e.d.

**Corollary.** *If  $g(H) = g(I) = -pI$  where  $H$  is unimodular but non-orthogonal, then  $p > 0$ .*

**Theorem 3.** *If there exists a local configuration  $M_o$  such that the corresponding stress  $S_o$  vanishes or is a negative hydrostatic pressure, then the material under consideration is a solid, and  $M_o$  is an undistorted configuration.*

The theorem states that if  $S_o = -pI$ , with  $p \leq 0$ , then the isotropy group  $\mathcal{I}$  relative to  $M_o$  is a subgroup of  $\mathcal{O}$ .

**Proof.** By the definition of  $\mathcal{I}$ , if  $H$  is in  $\mathcal{I}$ , then  $|\det H| = 1$  and  $g(H) = g(I) = S_o$ . Since we here have  $S_o = -pI$  with  $p \leq 0$ , the Corollary to Theorem 2 shows that  $H$  must be orthogonal, q.e.d.

Let us now consider elastic fluids. On turning back to Proposition 7 of Section 2 and observing that the stress in a fluid is characterized by a scalar-valued function  $p$  of a scalar  $\varrho$ , we see that our WTI is equivalent to our TI for a fluid.<sup>27</sup> It follows from Theorem 3 that  $p(\varrho)$  in (2.12) is always positive. Furthermore, it follows from Theorem 1 that  $p(\varrho)$  is an invertible function. These observations are sharpened in the following theorem.<sup>11</sup>

**Theorem 4.** *For an elastic fluid, a necessary and sufficient condition for the WTI (and hence also for the TI) is that the function  $p(\varrho)$  in (2.12) obey the following inequalities for all  $\varrho$ :*

$$p(\varrho) > 0, \quad (5.19)$$

$$\varrho \frac{d p(\varrho)}{d \varrho} \geq \frac{2}{3} p(\varrho). \quad (5.20)$$

In (5.20) it is to be understood that equality holds on only a nowhere dense set of values of  $\varrho$ .

Theorem 4 has the same content as Theorem 6 of reference 9, as is easily seen by observing that (5.20) is equivalent to the statement that the internal energy density in a fluid is a convex function of the cube root of the specific volume.<sup>28</sup>

Various implications of the WTI for isotropic materials are derived by COLEMAN & NOLL<sup>29</sup>, TRUESDELL & TOUPIN<sup>12</sup>, BRAGG & COLEMAN<sup>30</sup>, and NOLL & TRUESDELL<sup>31, 32</sup>.

<sup>27</sup> This is not the case when thermodynamic variables such as the temperature or entropy density are considered.

<sup>28</sup> An extension of the inequality (5.20) to a statement about the mean pressure on arbitrary states of strain in general elastic materials is given in reference 26.

<sup>29</sup> Section 12 of reference 9.

<sup>30</sup> BRAGG, L. E., & B. D. COLEMAN, J. Math. Phys. 4, 424 (1963).

<sup>31</sup> NOLL, W., & C. TRUESDELL, in: Encyclopedia of Physics, Edited by S. FLÜGGE, Vol. III/3. Berlin-Göttingen-Heidelberg: Springer (forthcoming).

<sup>32</sup> For its discussion of inequalities in isotropic materials the article of TRUESDELL & TOUPIN<sup>12</sup> is the most exhaustive currently in print. It also contains some interesting remarks on uniqueness questions for boundary value problems in materials of arbitrary symmetry.

### 6. Differential Inequalities

We assume now that the response function  $g$  of Eq. (2.1) or, equivalently, the response function  $\mathfrak{h}$  of Eq. (4.2) is continuously differentiable and investigate the restriction that the WTI imposes on the gradients of  $g$  and  $\mathfrak{h}$ .

Let  $D$  be an arbitrary non-zero symmetric tensor, and consider

$$U = I + \tau D, \quad (6.1)$$

where  $\tau$  is a real parameter  $\neq 0$ . It is clear that when  $|\tau|$  is sufficiently small the tensor  $U$  given by Eq. (6.1) is not only symmetric but also positive-definite and  $\neq I$ . Let us make the abbreviation

$$f(\tau) = \text{tr}\{D\mathfrak{h}(I + \tau D)^T\}. \quad (6.2)$$

Starting with (4.12) and letting  $U$  be given by (6.1), we find that the WTI may be written

$$\tau[f(\tau) - f(0)] > 0. \quad (6.3)$$

Dividing (6.3) by  $\tau^2$  and taking the limit  $\tau \rightarrow 0$ , we obtain

$$\dot{f}(0) \geq 0, \quad (6.4)$$

where the superimposed dot denotes the derivative. The directional derivative  $\nabla\mathfrak{h}[D]$  of  $\mathfrak{h}$  at  $I$ , in the direction of the tensor  $D$ , is defined by

$$\left. \frac{d}{d\tau} \mathfrak{h}(I + \tau D) \right|_{\tau=0} = \nabla\mathfrak{h}[D]. \quad (6.5)$$

Here  $\nabla\mathfrak{h}$ , the *gradient* of  $\mathfrak{h}$  at  $I$ , is a linear transformation of the six-dimensional space of symmetric tensors  $D$  into the nine-dimensional space of arbitrary tensors. We now define a quadratic form  $\mathfrak{Q}$  on the six-dimensional space of all symmetric tensors  $D$  by

$$\mathfrak{Q}(D) = \text{tr}\{D(\nabla\mathfrak{h}[D])^T\}. \quad (6.6)$$

Equations (6.2), (6.5), and (6.6) show that the inequality (6.4) states that  $\mathfrak{Q}(D) \geq 0$ . Thus we have

**Theorem 1.** *In order for the WTI to hold it is necessary that the quadratic form  $\mathfrak{Q}$  defined by (6.6) be positive-semidefinite, i.e. that*

$$\mathfrak{Q}(D) \geq 0 \quad (6.7)$$

for all symmetric tensors  $D$ .

Of course, the response function  $\mathfrak{h}$  and hence the quadratic form  $\mathfrak{Q}$  depend on the choice of the reference configuration;  $\mathfrak{Q}$  must be positive semi-definite for every such choice.

To cast (6.7) into a more transparent form we consider a smooth one-parameter family of deformation gradients  $F(\tau)$  such that  $F(0) = I$ . The corresponding rotation tensors are denoted by  $R(\tau)$  and the corresponding right stretch tensors by  $U(\tau)$ . We then have  $R(0) = U(0) = I$ . If we put

$$\dot{R}(0) = W, \quad \dot{U}(0) = D, \quad (6.8)$$

we find, by differentiating Eq. (4.7) with respect to  $\tau$  and then putting  $\tau = 0$ , that

$$\dot{F}(0) = W + D. \quad (6.9)$$

Moreover, it is clear that  $D$  is symmetric and it is easy to show that  $W$  is skew (by differentiating  $R^T R = I$ ).

Let  $S(\tau)$  and  $T(\tau)$  be the stress and the Piola-Kirchhoff tensor, respectively, corresponding to  $F(\tau)$ . Equation (4.1) shows that  $T(0) = S(0)$ . We use the notation

$$S(0) = T(0) = S_o, \quad \dot{S}(0) = \dot{S}_o, \quad \dot{T}(0) = \dot{T}_o. \quad (6.10)$$

On differentiating (4.5) we find

$$\dot{T}_o = W \mathfrak{h}(I) + V \mathfrak{h}[D], \quad (6.11)$$

where  $V \mathfrak{h}$  is the gradient of  $\mathfrak{h}$  at  $I$  and where the notations (6.9) and (6.10) have been used. Since  $S_o = \mathfrak{h}(I)$ , Eq. (6.11) is equivalent to

$$\dot{T}_o - W S_o = V \mathfrak{h}[D]. \quad (6.12)$$

Let us denote the density of the local configuration corresponding to  $F(\tau)$  by  $\varrho(\tau)$  and use the notation  $\varrho_o = \varrho(0)$ ,  $\dot{\varrho}_o = \dot{\varrho}(0)$ . If we differentiate Eq. (4.4) with respect to  $\tau$  and then put  $\tau = 0$ , we find, using (6.8)–(6.10),

$$\varrho_o \dot{S}_o = \dot{\varrho}_o S_o + \varrho_o S_o (D - W) + \varrho_o \dot{T}_o. \quad (6.13)$$

The equation of continuity<sup>33</sup> implies  $\dot{\varrho}_o = -(\text{tr} D) \varrho_o$ , hence (6.13) is equivalent to

$$\dot{S}_o = -S_o (\text{tr} D) + S_o (D - W) + \dot{T}_o. \quad (6.14)$$

Combining Eqs. (6.6), (6.12), and (6.14), we find

$$\mathbf{\Omega}(D) = \text{tr} \{ (\dot{S}_o + S_o W - W S_o) D \} + \text{tr} D \text{tr} (S_o D) - \text{tr} (S_o D^2). \quad (6.15)$$

If we differentiate Eq. (2.3) with respect to  $\tau$  and then put  $\tau = 0$ , we find

$$\dot{S}_o + S_o W - W S_o = V g[D], \quad (6.16)$$

where  $V g$  is the gradient of the response function  $g$  at the tensor  $I$ . We may regard  $V g$  as a linear transformation of the six-dimensional space of symmetric tensors into itself. When components are used, the matrix of  $V g$  has 36 components, which are the elastic coefficients for an infinitesimal deformation from the reference configuration. Of course, these elastic coefficients depend on the choice of the reference configuration.

It follows from Eqs. (6.15) and (6.16) that the quadratic form  $\mathbf{\Omega}(D)$  may be expressed in terms of  $V g$  by

$$\mathbf{\Omega}(D) = \text{tr} \{ D V g[D] \} + (\text{tr} D) \text{tr} (S_o D) - \text{tr} (S_o D^2). \quad (6.17)$$

We call a one-parameter family  $M = M(t)$  of local configurations a *deformation path*. If we define  $F(\tau)$  by  $M(t + \tau) = F(\tau) M(t)$ , the tensors  $D$  and  $W$  are called the *stretching* and the *spin* of the deformation path. In the notation of Eqs. (6.10)–(6.17) the stress corresponding to  $M(t)$  is  $S_o$ . We now omit the subscript  $o$  and note that Eq. (6.15) and Theorem 1 yield

**Theorem 2.** *In order for the WTI to hold it is necessary that for every deformation path*

$$\text{tr} (S^* D) \geq 0, \quad (6.18)$$

<sup>33</sup> Or, equivalently, (1.3).

where  $D$  is the stretching, and  $S^*$  is the invariant stress rate defined by

$$S^* = \dot{S} + SW - WS + S \operatorname{tr} D - \frac{1}{2}(SD + DS). \quad (6.19)$$

In (6.19)  $S$  is the stress and  $W$  the spin.

In his "Recherches sur l'élasticité," P. DUHEM<sup>34</sup> makes the following assertion.

"Considérons d'abord un système défini par un certain nombre de variables normales  $\alpha, \beta, \dots, \lambda$ , hors la température absolue  $T$ . Supposons qu'à une certaine température  $T$ , le système prenne un état d'équilibre lorsqu'on le soumet aux actions extérieures  $A, B, \dots, L$ , et que cet état d'équilibre varie d'une manière continue lorsque, sans faire varier la température  $T$ , on fait varier les valeurs  $A, B, \dots, L$  des actions extérieures. Si les actions  $A, B, \dots, L$  éprouvent des variations infiniment petites  $dA, dB, \dots, dL$ , que nous nommerons des actions perturbatrices, les valeurs des variables  $\alpha, \beta, \dots, \lambda$  qui conviennent à l'équilibre éprouvent des variations  $d\alpha, d\beta, \dots, d\lambda$ , que nous nommerons des perturbations; l'expression

$$dA d\alpha + dB d\beta + \dots + dL d\lambda$$

sera nommée le travail perturbateur isothermique.

"Dire qu'un travail perturbateur est positif, c'est dire, sous une forme mathématique précise, que la perturbation se produit dans le sens vers lequel tendent les actions perturbatrices. Il est clair que les systèmes que la nature nous offre seront tels, en général, que tout travail perturbateur isothermique, accompli à partir d'un état d'équilibre, soit positif. C'est ce que nous experimerons en disant qu'ils sont soumis à la loi du déplacement isothermique de l'équilibre."

Our Theorem 2 above furnishes a precise mathematization, for elastic materials, of DUHEM'S "law of isothermal displacement of equilibrium".

### Isotropy

We now seek the limitations which the WTI places on the elastic coefficients characterizing the response of an isotropic material to an infinitesimal deformation from an undistorted state.

Taking the gradient of Eq. (2.11) with respect to  $F$  at  $F=I$ , we find that

$$\nabla g[QDQ^{-1}] = Q \nabla g[D] Q^{-1} \quad (6.20)$$

holds for all orthogonal tensors  $Q$  in the isotropy group and all symmetric tensors  $D$ . Here we assume that  $g$  is the response function for an undistorted configuration of an isotropic material; it then follows that  $\nabla g[D]$  is a linear isotropic tensor function of  $D$ . It is a consequence of this remark that  $\nabla g$  has the representation<sup>35</sup>

$$\nabla g[D] = \lambda(\operatorname{tr} D)I + 2\mu D, \quad (6.21)$$

where  $\lambda$  and  $\mu$  (the Lamé coefficients) are elastic coefficients which depend on the reference configuration. By Proposition 6 of Section 2, the stress  $S_o$  on the undistorted reference configuration must be of the form

$$S_o = -pI. \quad (6.22)$$

Substitution of Eqs. (6.21) and (6.22) into (6.17) gives

$$\Omega(D) = (\lambda - p)(\operatorname{tr} D)^2 + (2\mu + p)\operatorname{tr} D^2. \quad (6.23)$$

<sup>34</sup> DUHEM, P., Ann. École Norm. 22, 143–217 (1905), p. 193.

<sup>35</sup> Cf. A. E. H. LOVE, A Treatise on the Mathematical Theory of Elasticity (Cambridge, 4th Edition, 1927), §§ 106–110. LOVE'S assumption of the existence of a strain energy function does not affect the generality of his results for isotropic materials or cubic crystals.

Now, quadratic forms of the type

$$q = a(d_1 + d_2 + d_3)^2 + b(d_1^2 + d_2^2 + d_3^2) \quad (6.24)$$

are well known. A necessary and sufficient condition that  $q$  be positive semi-definite in the triple  $(d_1, d_2, d_3)$  is that both

$$b \geq 0 \quad \text{and} \quad 3a + 2b \geq 0. \quad (6.25)$$

If we take  $a = \lambda - \phi$ ,  $b = 2\mu + \phi$ , and  $d_i$  to be the proper numbers of  $D$ , the quadratic form (6.24) reduces to (6.23). Therefore, a necessary and sufficient condition that  $\Omega(D)$  be positive semi-definite is that

$$2\mu + \phi \geq 0 \quad \text{and} \quad 3(\lambda - \phi) + 2(2\mu + \phi) \geq 0. \quad (6.26)$$

From Theorem 1 we now obtain the following result.

**Theorem 3.** *The WTI implies that for each undistorted configuration of an isotropic material the following inequalities hold:*

$$\mu \geq -\frac{1}{2}\phi, \quad (6.27a)$$

$$\kappa \geq \frac{2}{3}\phi; \quad (6.27b)$$

here  $\phi$  is the pressure,  $\mu$  the shear modulus, and  $\kappa = \lambda + \frac{2}{3}\mu$  the compression modulus; of course  $\phi$ ,  $\mu$ , and  $\kappa$  can all depend on the choice of the reference configuration.

If  $\phi = 0$ , i.e., if the reference configuration is a natural state of an isotropic solid, then Theorem 3 yields the familiar assertion that

$$\mu \geq 0 \quad \text{and} \quad \kappa \geq 0. \quad (6.28)$$

For an elastic fluid we have  $\mu = 0$  and  $\kappa = \rho d\phi(\rho)/d\rho$ , and hence Theorem 3 yields

$$\phi \geq 0, \quad \rho \frac{d\phi(\rho)}{d\rho} \geq \frac{2}{3}\phi.$$

This condition is just a bit weaker than the necessary and sufficient condition for the WTI stated in Theorem 4 of Section 5, for our present Theorem 3 does not contain the precious result that  $\phi = 0$  is impossible in a fluid. Yet we think Theorem 3 to be not without interest. Through its conclusion (6.27a) it yields a condition on  $\mu$  for a particular negative pressure  $\phi$  to be possible: If the configuration is varied among undistorted configurations in such a way that  $\phi$  decreases and becomes negative,  $\mu$  must always remain greater than  $-\frac{1}{2}\phi$ , otherwise the material will not support the negative pressure.<sup>36</sup>

### Cubic Symmetry

When the material has cubic symmetry and the reference configuration is undistorted, it follows from (6.20) that there exists an orthonormal basis  $\mathbf{n}_i$  such that the components of  $\nabla g[D]$  and  $D$  are related by<sup>37</sup>

$$\nabla g[D]_{i,i} = \alpha D_{i,i} + \beta(D_{j,j} + D_{k,k}), \quad (6.29a)$$

$$\nabla g[D]_{i,j} = 2\gamma D_{i,j}. \quad (6.29b)$$

<sup>36</sup> We take this to mean that cavitation will occur, as seems to be the case with fluids.

<sup>37</sup> The result is essentially given in LOVE's treatise.<sup>35</sup> The three coefficients  $\alpha, \beta, \gamma$ , are related to those on p. 160 of reference 35 by  $C_{11} = \alpha$ ,  $C_{12} = \beta$ ,  $C_{44} = \gamma$ .



No summation convention is used here, and  $i, j, k$  is any permutation of 1, 2, 3. It follows from the last entry in Table 2 that the stress  $S_o$  in the reference configuration must be a hydrostatic pressure:

$$S_o = -pI. \tag{6.30}$$

Substitution of (6.29) and (6.30) into (6.17) yields

$$\Omega(D) = \alpha \sum_{i=1}^3 D_{ii}^2 + 2(\beta - p) \sum_{\substack{i,j=1 \\ j>i}}^3 D_{ii} D_{jj} + (4\gamma + 2p) \sum_{\substack{i,j=1 \\ j>i}}^3 D_{ij}^2. \tag{6.31}$$

Thus  $\Omega(D)$  is the sum of the two quadratic forms

$$\Omega_1(D) = \alpha \sum_{i=1}^3 D_{ii}^2 + (2\beta - p) \sum_{\substack{i,j=1 \\ j>i}}^3 D_{ii} D_{jj}, \tag{6.32 a}$$

and

$$\Omega_2(D) = (4\gamma + 2p) \sum_{\substack{i,j=1 \\ j>i}}^3 D_{ij}^2. \tag{6.32 b}$$

The form  $\Omega_1(D)$  involves the three diagonal components of  $D$ , which can be chosen at will, while the form  $\Omega_2(D)$  involves only the three off-diagonal components of  $D$ , which also can be chosen at will. Hence, in order for  $\Omega(D)$  to be positive-semidefinite it is necessary and sufficient that both of the forms (6.32) be positive-semidefinite.

It is clear that the form  $\Omega_2(D)$  of (6.32b) is positive-semidefinite if and only if  $4\gamma + 2p \geq 0$ . The form  $\Omega_1(D)$  of (6.32a) is of the type (6.24) when we put  $a + b = \alpha$ ,  $2a = 2\beta - p$ , and  $d_i = D_{ii}$ ,  $i = 1, 2, 3$ . Using (6.25), it follows that the form  $\Omega(D)$  of (6.31) is positive-semidefinite if and only if the three inequalities

$$2\gamma + p \geq 0, \quad \alpha - \beta + \frac{1}{2}p \geq 0, \quad \beta + 2\alpha - \frac{1}{2}p \geq 0 \tag{6.33}$$

hold. We summarize:

**Theorem 4.** *The WTI implies that for each undistorted configuration of a material with cubic symmetry, the elastic coefficients  $\alpha, \beta, \gamma$  and the pressure  $p$  obey the inequalities*

$$\gamma \geq -\frac{1}{2}p, \quad \alpha - \beta \geq -\frac{1}{2}p, \quad \beta + 2\alpha \geq \frac{1}{2}p. \tag{6.34}$$

*Acknowledgement.* This research was supported by the Air Force Office of Scientific Research under Contract AF 49(638)541 with Mellon Institute and by the National Science Foundation under Grant NSF-G 16745 to Carnegie Institute of Technology.

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(Received August 16, 1963)