The Derivation of Constitutive Relations from the Free Energy and the Dissipation Function

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I. Introduction

Mechanics may be based on two principles: the reaction principle and the principle of virtual power. The reaction principle (still cited in most books in its special Newtonian version, valid merely for pairs of material points) must be stated in the following general form: (1) Every force has its reaction. A force is called internal or external for a given system according to whether its reaction acts within it or without. (2) In any virtual state of motion corresponding to a rigid displacement of the system, the total power of the internal forces is zero. The principle of virtual power states that in any virtual state of motion (whether admissible or not) the total power of the internal, external, and inertia forces is zero.

Applying these two principles to translations and rotations (particular cases of rigid motions), one obtains the theorems of linear and angular momentum. They are free of internal forces.

In the case of a discrete system, composed of material points and/or rigid bodies, certain internal forces are usually known, whereas others (called reactions) are unknown. Applying the theorems of linear and angular momentum to the entire system, we may not obtain sufficient differential equations to determine its motion. However, if we apply the theorems to the various parts of the given system, obtained by successive subdivision, more and more internal forces become external, and the same process increases the number of differential equations until a complete set is obtained to determine the motion and the reactions. Integration of this set may be difficult. It can be facilitated, however, by applying the principle of virtual power to the real motion and to the most general admissible state of motion.

183

In this manner one obtains, respectively, the energy theorem and Lagrange's equations.

Let us note in passing that to formulate Lagrange's equations (and in consequence analytical mechanics as a whole), one starts by choosing a set of generalized coordinates q_i, together with the corresponding velocities $\dot{q}_i$, and proceeds to define the generalized forces Q_i by means of the expression for the virtual work or power. There is no way to invert this process: velocities and forces are inexchangeable.

In the case of a continuum the system to be considered is the mass element, and the internal forces are by definition unknown. In place of the q_i we have the coordinates determining position and orientation of the element plus a set of strains determining its state of deformation. In place of the $\dot{q}_i$ we have the linear and angular velocities of the element plus the material strain rates, and the Q_i appear replaced by the resultant force and moment plus the stresses defined by the expression for the virtual work or power of deformation. The theorems of linear and angular momentum are still valid; however, they are insufficient to determine the motion. Since the internal forces are unknown, so is the connection between strains and stresses. Subdivision of the element is of no avail since it procuces elements with the properties of the orignal one unless the subdivision is carried to the point where the continuum loses its character and disintegrates into its molecules.

There are two ways out of this impasse. The first has been followed in the past by continuum mechanics. Here, the observed response of given materials is modeled by constitutive relations connecting the stresses with the strains, the strain rates, or even the strain history. In cases where the history comes into play, an alternative—which will be preferred in this article—can be based on the introduction of internal parameters in the form of strains, in the definition of the corresponding internal stresses and in the inclusion of these entities in the constitutive relations. These approaches have been quite successful. However, they are semiempirical, different for each material, and not tied together by a general principle.

The other way out of the dilemma consists in the attempt to explain the macroscopic response of a material in terms of its microscopic structure. Purely mechanical efforts in this direction have not been very successful. However, the example set by the kinetic theory of gases appears promising. Here, the thermodynamic behavior of the macrosystem (the mass element of the continuum) is explained by the mechanics of the microsystem (the total of the particles of which the element is composed). Since the microscopic motion eludes the macroscopic observer in its details, statistics must be used to interpret it on the phenomenological level. As the particles are in continuous motion, the temperature has to be introduced as a macroscopic measure of the microscopic kinetic energy, and the energy flux by microscopic exchange between particles within and outside the element while it

retains its shape must be interpreted phenomenologically as a heat flow. In short, the attempt to physically understand material response turns continuum mechanics into thermodynamics.

The reverse is true as well. As long as thermodynamics confined itself to gases, its connection with continuum mechanics remained concealed. Once it is applied to other materials, the molar volume has to be replaced by the strains and the pressure by stresses, and it becomes clear that thermodynamics and continuum mechanics become inseparable, forming one single branch of science.

As a first consequence of this insight, the mechanical energy theorem is to be dropped. True, it still holds for the microsystem, but on the phenomenological level it must be replaced by the first fundamental law of thermodynamics. The next consequence is the recognition of the second fundamental law. Moreover, it is imperative that both laws be applied not to the entire continuum, but to its material elements. In other words, thermodynamics is to be conceived as a field theory in the same manner as continuum mechanics. Once this is recognized, the time-honored restriction to extremely slow processes becomes obsolete, and it becomes possible to treat reversible as well as irreversible processes with the same ease. It is clear, on the other hand, that exceedingly fast processes, where the macro- and microvelocities become comparable, have to be avoided, but then they are exceptions within the framework of continuum mechanics.

In the vast majority of cases, deformation of a continuum is an irreversible process. So is heat conduction. Here, the fundamental laws are not sufficient to establish constitutive relations. The gap has been partially closed by Onsager (1931) with his reciprocity relations. They establish the symmetry of the matrix connecting velocities and forces. In this function they are clearly restricted to linear processes and hence lack the status of a physical law, not to mention the fact that many processes in continuum mechanics are nonlinear.

About a generation ago one of the present writers (Ziegler, 1958) proposed an orthogonality principle for irreversible processes, based on the dissipation function and including Onsager's theory as a special case. He later showed (Ziegler, 1961) that orthogonality (which, incidentally, requires regular or at least regularized dissipation functions) is equivalent to a number of extremum principles. The most appealing of these (applicable also in cases of irregular dissipation functions) postulates that the irreversible process, subject to certain side conditions, always maximize the rate of entropy production. During the ensuing decades these principles were discussed in various publications and restricted to purely dissipative processes of the elementary and the complex types. (The first restriction precludes the presence of gyroscopic forces such as Coriolis forces or magnetic fields. Elementary processes are defined by velocities in the form of components

of a single vector or tensor; in the complex case more elementary processes may occur at the same time but must be coupled.)

Continuum mechanics has proved to be the most fertile field of application of the principles mentioned. Here, they allow one to establish constitutive relations, deducing them from a single pair of scalar functions characterizing the material: its free energy and its idssipation function. The process and some of its results have been described in the first writer's book on thermomechanics (Ziegler, 1977), and some questions still open in the first edition have been elaborated in the second (Ziegler, 1983). In the meantime it has been possible to generalize and supplement the few applications treated in this book. Besides, more applications have been supplied by Houlsby (1979, 1980, 1981a,b) and by Germain *et al.* (1983).

The writers feel that the time has come to present a systematic account of the leading functions providing constitutive relations. Such an account is the topic of this article. The principle of maximal rate of entropy production is quite general and can be applied with the same ease to small and large deformations. In spite of this, the review will be restricted, wherever the strain tensor becomes part of the development, to its first approximation, that is, to infinitesimal strains. The authors hope that the results may motivate future work extending the method to finite deformations.

The next section presents a brief account of the theory. For its justification and for more details the reader is referred to the book (Ziegler, 1983), which will be cited for brevity as Z, followed by the number of the relevant section. Some of the mathematical tools needed are collected in an appendix at the end of the article. The rest of the article is devoted to a variety of materials. It will show that the response of most of the models in use can in fact be derived from their free energy and their dissipation function. It will also show that (and why) a few models do not fit into the pattern.

II. Thermomechanical Theory

A. THERMODYNAMICS

Let us consider a mass element of a continuum (Z.4.3), and let us provisionally assume that its state can be described by the absolute temperature $\vartheta > 0$ and a set of mechanical parameters, the components of a strain tensor ε_{ij}. Let us further assume that σ^{ij} is a stress tensor and that the expression

$$l := (1/\rho)\sigma^{ij}\dot{\varepsilon}_{ij}, \qquad (2.1)$$

where ρ is a density, the dot indicates material differentiation, and the summation convention is used, representing the specific power of deformation.

Here and in what follows, "specific" will always mean "per unit mass." In an expression like (2.1) it is customary in thermodynamics to refer to the $\dot{\varepsilon}_{ij}$ as velocities and to the σ^{ij}/ρ as the associated (specific) forces.

Macvean (1968) has studied the various strain tensors proposed in the past. He has shown that only a few of them allow a representation (2.1) of the specific power and thus are thermodynamically acceptable. One of them is Green's strain tensor, based on the so-called Lagrangean description of the deformation; it is associated with Kirchhoff's stress tensor, and ρ has to be interpreted as the density in the reference state.

It will be simpler for our purpose to use the so-called Eulerian approach, based on rectangular Cartesian coordinates, and to replace (2.1) by

$$l = (1/\rho)\sigma_{ij}d_{ij},\tag{2.2}$$

where ρ is the instantaneous density; σ_{ij} is Euler's stress tensor; and $d_{ij} = (v_{j,i} + v_{i,j})/2$ is the deformation rate in the sense of Prager (1961), with subscripts following a comma denoting partial differentiation with respect to the corresponding coordinate so that $v_{j,i}$ is the velocity gradient. It is true that d_{ij} cannot generally be interpreted as the time derivative of a strain. However, if we restrict ourselves, where necessary, to small deformations, we may write $d_{ij} = \dot{\varepsilon}_{ij}$, where ε_{ij} is the engineering strain and σ_{ij} the engineering stress, the dot indicates partial differentiation with respect to time, and ρ may be considered constant.

The entities ε_{ij} and ϑ will be considered as independent state variables. Functions of them are dependent state variables, also denoted as state functions. Examples are the specific internal energy $u(\varepsilon_{ij}, \vartheta)$, the specific entropy $s(\varepsilon_{ij}, \vartheta)$, and the specific free energy

$$\psi(\varepsilon_{ij}, \vartheta) := u - \vartheta s.\tag{2.3}$$

If we exclude radiation, the specific heat supply per unit time is $-q_{i,i}/\rho$, where q_i is the vector of heat flow. The specific entropy supply per unit time is $-(q_i/\vartheta)_{,i}/\rho$.

The first fundamental law

$$\dot{u} = (1/\rho)(\sigma_{ij}d_{ij} - q_{i,i})\tag{2.4}$$

states that the rate of increase of the specific internal energy is the sum of the specific power of the stresses and the specific rate of heat supply. The second fundamental law may be written

$$\dot{s} = s^{*(r)} + s^{*(i)},\tag{2.5}$$

where

$$s^{*(r)} = -(1/\rho)(q_i/\vartheta)_{,i} \qquad \text{and} \qquad s^{*(i)} \geqq 0.\tag{2.6}$$

It states that the specific rate of entropy increase consists of a reversible and an irreversible contribution. The reversible contribution is the entropy

supply from outside; the irreversible contribution, to be interpreted as an entropy production within the element, is nonnegative.

From (2.4) through (2.6) we deduce

$$\rho(\dot{u} - \vartheta\dot{s}) = \sigma_{ij}d_{ij} - \frac{q_i}{\vartheta}\vartheta_{,i} - \rho\vartheta s^{*(i)} \qquad (2.7)$$

or

$$\sigma_{ij}d_{ij} = \rho\left(\frac{\partial u}{\partial\varepsilon_{ij}} - \vartheta\frac{\partial s}{\partial\varepsilon_{ij}}\right)d_{ij} + \rho\left(\frac{\partial u}{\partial\vartheta} - \vartheta\frac{\partial s}{\partial\vartheta}\right)\dot{\vartheta} + \frac{q_i}{\vartheta}\vartheta_{,i} + \rho\vartheta s^{*(i)}. \quad (2.8)$$

In the special case of pure heating (or cooling) the rate of deformation is zero and the element receives (or loses) heat, whereas no net flow of heat across the element takes place. Thus, $d_{ij} = 0$ and $q_i = 0$ but $q_{i,i} \neq 0$. Since the process is reversible and $\dot{\vartheta}$ is different from zero, it follows from (2.8) that

$$\partial u/\partial\vartheta = \vartheta(\partial s/\partial\vartheta), \qquad (2.9)$$

and since both sides of (2.9) are state functions and hence are independent of the particular process, this equation must be generally valid. Thus (2.8) reduces to

$$\sigma_{ij}d_{ij} = \rho(\partial\psi/\partial\varepsilon_{ij})d_{ij} + (q_i/\vartheta)\vartheta_{,i} + \rho\vartheta s^{*(i)}, \qquad (2.10)$$

where (2.3) has been used.

Equation (2.10) suggests the subdivision of the stress tensor into two parts,

$$\sigma_{ij} = \sigma_{ij}^{(q)} + \sigma_{ij}^{(d)}, \qquad (2.11)$$

in such a way that

$$\sigma_{ij}^{(q)} = \rho(\partial\psi/\partial\varepsilon_{ij}). \qquad (2.12)$$

Since the $\sigma_{ij}^{(q)}$ defined in this manner depend on a potential ψ, which, however, is a function of the ε_{ij} *and* ϑ, we call them quasiconservative stresses, and we note that, on account of (2.3) and (2.9), Eq. (2.12) may be supplemented by

$$s = -\partial\psi/\partial\vartheta. \qquad (2.13)$$

For the remainders $\sigma_{ij}^{(d)}$ in (2.11), Eq. (2.10), together with the inequality (2.6), supplies the condition

$$\sigma_{ij}^{(d)}d_{ij} - (\vartheta_{,i}/\vartheta)q_i = \rho\vartheta s^{*(i)} \geqq 0, \qquad (2.14)$$

that is, the so-called Clausius–Duhem inequality. A glance at (2.2) shows that the $\sigma_{ij}^{(d)}$ correspond to specific forces $\sigma_{ij}^{(d)}/\rho$, which appear in irreversible deformations; they will be denoted as dissipative stresses. Heat flow q_i *across* the element (in contrast to heat supply, determined by $q_{i,i}$) is another irreversible process; the corresponding force is $-\vartheta_{,i}/\rho\vartheta$.

The case just discussed is particularly simple. In addition to ε_{ij} and ϑ, most materials also require internal parameters for the description of their state. Let us assume, for example, that a single internal strain tensor α_{ij} is needed, and let us denote the associated force by β_{ij}/ρ. Independent state variables are then ε_{ij}, α_{ij}, ϑ; the state functions depend on all of them, and the terms with d_{ij} in (2.7) through (2.10) must be supplemented by the corresponding terms with $\dot{\alpha}_{ij}$. Thus, (2.12) is to be supplemented by

$$\beta_{ij}^{(q)} = \rho(\partial\psi/\partial\alpha_{ij}), \tag{2.15}$$

and (2.14) must be replaced by

$$\sigma_{ij}^{(d)}d_{ij} + \beta_{ij}^{(d)}\dot{\alpha}_{ij} - (\vartheta_{,i}/\vartheta)q_i = \rho\vartheta s^{*(i)} \geqq 0. \tag{2.16}$$

However, since the α_{ij} are internal parameters, the β_{ij} do not appear in the first fundamental law, and if we exclude gyroscopic components (i.e., forces that depend on the velocities $\dot{\alpha}_{ij}$ in such a manner that their power is always zero, as, e.g., the Lorentz force acting in a magnetic field or the Coriolis force in a rotating coordinate system), we have $\beta_{ij} = 0$ or equivalently

$$\beta_{ij}^{(q)} + \beta_{ij}^{(d)} = 0. \tag{2.17}$$

The generalization for more than one internal strain tensor is obvious.

B. Orthogonality

The simplest materials dealt with in continuum mechanics are elastic. They may be defined (Z.5.1) by the conditions $\alpha_{ij} = 0$ and $\sigma_{ij}^{(d)} = 0$. Thus, the first two terms in (2.16) are zero. In the isothermal or adiabatic case, the last term vanishes, too, and the deformation becomes reversible process. It is entirely determined by the specific free energy $\psi(\varepsilon_{ij})$: the specific entropy follows from (2.13), the internal energy subsequently from (2.3), and the only constitutive equation (2.12) connects the stresses with the strains.

More general processes and those taking place in more general materials are irreversible and require more constitutive relations, connecting the dissipative forces $\sigma_{ij}^{(d)}/\rho$, $\beta_{ij}^{(d)}/\rho$, and $-\vartheta_{,i}/\rho\vartheta$ with the velocities d_{ij}, $\dot{\alpha}_{ij}$, and q_i. So far, the only condition at our disposal is (2.16),

$$\sigma_{ij}^{(d)}d_{ij} + \beta_{ij}^{(d)}\dot{\alpha}_{ij} - (\vartheta_{,i}/\vartheta)q_i = \rho\varphi \geqq 0, \tag{2.18}$$

where φ is the specific dissipation function,

$$\varphi := \vartheta s^{*(i)} \geqq 0, \tag{2.19}$$

which may be considered as a function of d_{ij}, $\dot{\alpha}_{ij}$, q_i and generally also of the state ε_{ij}, α_{ij}, ϑ.

The constitutive equations we are looking for have been established separately for many idealized materials, and the corresponding dissipation functions are easily derived from them. As stated in Section I, however, these relations are not connected by a general law, as should be expected, and attempts to deduce them mechanically from the microstructure of the various materials have not been very successful. Once the microstructure becomes important, however, it is inevitable that statics and hence thermodynamics come into play, and the mechanical reasoning just described can in fact be replaced by an approach that is more in the spirit of thermodynamics.

Let us note that, in the chain of reasoning leading from the fundamental laws to the inequality (2.14), the dissipation function (2.19) appeared already in (2.7), long before the dissipative stresses were introduced in order to interpret φ as a specific power. It therefore seems reasonable to invert (Z.14.2) the conventional approach: instead of starting from the forces and interpeting φ as a special expression of the specific power and hence as a function of secondary importance, we accept φ as the primary function and propose to derive the dissipative forces from it. This is equivalent to postulating (as already at the end of Section II,A) that the forces have no gyroscopic components. If it is possible to deduce the dissipative forces from φ in a similar manner as the quasi-conservative forces from ψ, each material is characterized by two scalar functions ψ and φ.

In the case of pure heat conduction, the velocities q_i are the components of a vector. The same is true for the dissipative forces $F_i^{(d)} := -\vartheta_{,i}/\rho\vartheta$. The dissipation function $\varphi(q_i)$, assumed to be regular, may be described in velocity space by the dissipation surfaces $\varphi = \text{const.}$ (Fig. 1). Equation (2.18) reduces to

$$F_i^{(d)}q_i = \varphi(q_i). \tag{2.20}$$

It determines the projection of the vector $F_i^{(d)}$ onto q_i, leaving, however, its direction free. If this direction is to be given by φ, it must, as has been shown in (Z.14.3), be determined by the gradient $\partial\varphi/\partial q_i$ of φ in velocity space and hence by

$$F_i^{(d)} = \nu(\partial\varphi/\partial q_i), \tag{2.21}$$

where $\nu = \varphi[(\partial\varphi/\partial q_k)q_k]^{-1}$ ensures that (2.20) is satisfied.
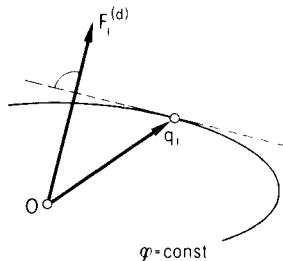

FIG. 1.   Orthogonality in velocity space.

Geometrically, Eq. (2.21) may be interpreted as an *orthogonality condition*: the dissipative force $F_i^{(d)}$ corresponding to a velocity q_i is orthogonal to the dissipation surface $\varphi = $ const passing through the end point of q_i.

Equation (2.21) maps the velocity space onto the space of the dissipative forces. If this mapping and its inverse are single valued, it can be shown (Z.14.5) that the dissipation surfaces are star shaped with respect to the origin, strongly convex, and ordered in the sense that each of them encloses those with smaller values of φ. It can even be demonstrated (Z.15.3) that the orthogonality condition implies that both sides of (2.20) are nonnegative, so that the second fundamental law is automatically satisfied and the vector $F_i^{(d)}$ has the direction of the outward normal of the dissipation surface.

All these results remain valid in the case of pure deformation without internal parameters or heat flow. Here, the velocities d_{ij} are the components of a symmetric tensor and so are the dissipative forces $F_{ij}^{(d)} := \sigma_{ij}^{(d)}/\rho$. Both tensors may be represented as vectors in a six-dimensional space, together with the dissipation surfaces $\varphi(d_{ij}) = $ const. In place of (2.20) and (2.21) we now have

$$F_{ij}^{(d)} d_{ij} = \varphi(d_{ij}) \qquad (2.22)$$

and

$$F_{ij}^{(d)} = \nu(\partial\varphi/\partial d_{ij}), \qquad (2.23)$$

where

$$\nu = \varphi[(\partial\varphi/\partial d_{kl})\, d_{kl}]^{-1}.$$

The dissipation surfaces have the same properties as in the case of heat conduction. Certain cases where φ is not regular will be treated in Sections VI and VII.

If, finally, the $\dot{\alpha}_{ij}$ are the only velocities—that is, in cases where heat flow and deformation are absent, whereas the internal parameters are still in the process of approaching their equilibrium values—the dissipative forces are $F_{ij}^{(d)} := \beta_{ij}^{(d)}/\rho$, and (2.22), (2.23) hold again provided that d_{ij} is replaced by $\dot{\alpha}_{ij}$.

C. Maximal Rate of Entropy Production

In order to discuss a few more general results, let us denote the various velocities considered in Section II,B by u_i and the dissipative forces by $F_i^{(d)}$, where the number of velocity components is three or six according to whether u_i is a vector or a symmetric tensor.

It has been mentioned in Section II,B that, in general, the dissipation function depends not only on the velocities, but also on the independent state variables. If it is a function of the velocities alone, the results obtained in velocity space have their corollaries in force space. Let $\varphi'(F_i^{(d)})$ be the

dissipation function in terms of the dissipative forces. Then, the corollary
(Z.14.3) of the orthogonality condition states that the velocity u_i correspond-
ing to a dissipative force $F_i^{(d)}$ is orthogonal to the dissipation surface
$\varphi' = \text{const}$ passing through the end point of $F_i^{(d)}$. A particularly interesting
example is the theory of the plastic potential (Section VI,A).

If, on the other hand, the dissipation function φ also depends explicitly
on the independent state variables, the corollary need not hold. In soil
mechanics, fro example, the theory of the plastic potential breaks down
(Section VII).

It can be shown (Z.15.1) that the orthogonality condition is equivalent
to various extremum principles. Of particular interest is the *principle of
maximal dissipation rate*: Provided that the dissipative force $F_i^{(d)}$ is pre-
scribed, the actual velocity u_i maximizes the dissipation rate $l^{(d)} = F_i^{(d)} u_i$
subject to the side condition

$$\varphi(u_i) = F_i^{(d)} u_i = l^{(d)} > 0. \tag{2.24}$$

It is obvious on account of (2.19) that this principle may also be stated
as a *principle of maximal rate of entropy production*: Provided that the
dissipative force $F_i^{(d)}$ is prescribed, the actual velocity u_i maximizes the
rate of entropy production $s^{*(i)}$ subject to the side condition

$$s^{*(i)} = (1/\vartheta) F_i^{(d)} u_i > 0. \tag{2.25}$$

We note that these extremum principles are slightly more general than
the orthogonality condition: they are still applicable in cases where φ
exhibits irregularities corresponding to singularities like edges or corners
in the dissipation surfaces. To treat such cases by means of orthogonality,
edges or corners have to be smoothened; that is, the singularities must be
considered as limiting cases of regular dissipation functions.

It often happens that a dissipation function is quasi-homogeneous, satisfy-
ing a functional equation of the type

$$(\partial \varphi / \partial u_i) u_i = F(\varphi), \tag{2.26}$$

where the function $F(\varphi)$ is free except for the condition $F(0) = 0$. The
corresponding dissipation surfaces are similar and similarly located with
respect to the origin. Ziegler (1963) has proved that in this special case the
dissipative forces may be derived from a potential, which is constant on
dissipation surfaces in velocity space. It is easy to show that the inversion
is true: representation of the dissipative forces by means of a potential
requires that this potential (and hence the dissipation function) be quasi-
homogeneous. This result, together with mathematical contributions by
Moreau (1970), gave rise to a theory of "pseudo-potentials" by Germain
(1973), which includes singularities like nonexistence of the gradient $\partial \varphi / \partial u_i$
for certain velocities (but excludes dissipation functions dependent on
velocities *and* explicitly also on the independent-state variables). Since

quasi-homogeneous dissipation functions are special cases and since their potential has no physical significance, we will not pursue this line of thought.

In (Z.14.3 and Z.15.1) the orthogonality condition and the equivalent extremum principles have been established for velocities in the form of vectors or symmetric tensors. The corresponding applications in Section II,B were heat low, deformation, and relaxation of the internal parameters, each of these processes considered separately. In general, they occur simultaneously, and the question arises how the orthogonality condition or the extremum principles are to be applied in this case. We obviously have two options (Z.14.4).

Case (a): If the various processes are independent, each of them is governed by its own orthogonality condition. There are three dissipation functions, $\varphi^{(1)}(d_{ij})$, $\varphi^{(2)}(\dot{\alpha}_{ij})$, and $\varphi^{(3)}(q_i)$. Each of them may also depend on ε_{ij}, α_{ij}, and ϑ. Using (2.21), (2.23) and inserting the proper dissipative forces, we obtain the constitutive equations

$$\sigma_{ij}^{(d)} = \nu^{(1)}\rho\frac{\partial\varphi^{(1)}}{\partial d_{ij}}, \qquad \beta_{ij}^{(d)} = \nu^{(2)}\rho\frac{\partial\varphi^{(2)}}{\partial\dot{\alpha}_{ij}}, \qquad -\frac{\vartheta_{,i}}{\vartheta} = \nu^{(3)}\rho\frac{\partial\varphi^{(3)}}{\partial q_i}, \quad (2.27)$$

where

$$\nu^{(1)} = \varphi^{(1)}\left(\frac{\partial\varphi^{(1)}}{\partial d_{kl}}d_{kl}\right)^{-1}, \qquad \nu^{(2)} = \varphi^{(2)}\left(\frac{\partial\varphi^{(2)}}{\partial\dot{\alpha}_{kl}}\dot{\alpha}_{kl}\right)^{-1},$$

$$\nu^{(3)} = \varphi^{(3)}\left(\frac{\partial\varphi^{(3)}}{\partial q_k}q_k\right)^{-1}. \qquad (2.28)$$

Since the entire dissipation rate is given by

$$\varphi(d_{ij}, \dot{\alpha}_{ij}, q_i) = \varphi^{(1)}(d_{ij}) + \varphi^{(2)}(\dot{\alpha}_{ij}) + \varphi^{(3)}(q_i), \qquad (2.29)$$

the superscripts of φ might be dropped in (2.27) but not in (2.28).

Case (b): If, on the other hand, the various processes are coupled, we have no means of establishing the constitutive relations unless we postulate that the orthogonality condition holds in the 15-dimensional space of all velocities d_{ij}, $\dot{\alpha}_{ij}$, q_i. The decomposition (2.29) does not hold, and (2.27) has to be replaced by

$$\sigma_{ij}^{(d)} = \nu\rho\frac{\partial\varphi}{\partial d_{ij}}, \qquad \beta_{ij}^{(d)} = \nu\rho\frac{\partial\varphi}{\partial\dot{\alpha}_{ij}}, \qquad -\frac{\vartheta_{,i}}{\vartheta} = \nu\rho\frac{\partial\varphi}{\partial q_i}, \qquad (2.30)$$

where

$$\nu = \varphi\left(\frac{\partial\varphi}{\partial d_{kl}}d_{kl} + \frac{\partial\varphi}{\partial\dot{\alpha}_{kl}}\dot{\alpha}_{kl} + \frac{\partial\varphi}{\partial q_k}q_k\right)^{-1}. \qquad (2.31)$$

Cases (a) and (b) confront us with an alternative. As observed in the conclusion of (Z), there is no continuous transition between them. In either case, the orthogonality condition ensures (see Z.15.3) that the Clausius-Duhem inequality (2.16) is satisfied; the second law thus appears as a

consequence of the principle of maximal rate of entropy production. With our present knowledge there is no way to decide between the possibilities (a) and (b).

The generalization of (2.27) through (2.31) for more internal strains is obvious. Incidentally, in the linear case, where φ is purely quadratic in the velocities and hence homogeneous of degree 2, all of the ν's are $\frac{1}{2}$, and one does not make a mistake using (2.30) in place of (2.27). In the applications to be discussed in the next few sections, it is not necessary yet to distinguish between cases (a) and (b). Later applications will be treated both ways or, where this does not seem necessary, according to case (a).

In order to obtain the entire stresses (2.11),

$$\sigma_{ij} = \sigma_{ij}^{(q)} + \sigma_{ij}^{(d)}, \tag{2.32}$$

the dissipative stresses just discussed have to be supplemented by the quasi-conservative stresses (2.12),

$$\sigma_{ij}^{(q)} = \rho(\partial\psi/\partial\varepsilon_{ij}), \tag{2.33}$$

where ψ is now a function of ε_{ij}, α_{ij}, and ϑ. Similarly, we have (2.15),

$$\beta_{ij}^{(q)} = \rho(\partial\psi/\partial\alpha_{ij}) \tag{2.34}$$

and, according to (2.17),

$$\beta_{ij}^{(d)} = -\beta_{ij}^{(q)}, \tag{2.35}$$

Relations (2.27) through (2.35) permit us to derive the constitutive equations (or inequalities) of any material from its specific free energy ψ (ε_{ij}, α_{ij}, ϑ) and its specific dissipation function φ (d_{ij}, $\dot{\alpha}_{ij}$, q_i, ε_{ij}, α_{ij}, ϑ). In many applications, particularly those where ρ may be considered constant, it is convenient to replace the functions ψ and φ by

$$\Psi = \rho\psi, \qquad \Phi = \rho\varphi, \tag{2.36}$$

that is, by the free energy and the dissipation function per unit volume.

III. Heat Conduction

Heat conduction without deformation is a purely dissipative process. Its velocity is the heat flow vector q_i; the corresponding dissipative force per unit volume is

$$\rho F_i^{(d)} = -\vartheta_{,i}/\vartheta. \tag{3.1}$$

Since the quasi-conservative force is identically zero, and the free energy is constant and may be assumed to be naught. As regards the dissipation function, it is convenient to set the factor $1/\vartheta$ in evidence. Thus, the most

general case is governed by the functions

$$\Psi = 0, \qquad \Phi = \rho\varphi = (1/\vartheta)\gamma(q_i), \tag{3.2}$$

where $\gamma(q_i)$ is a postiive definite function, possibly also dependent on ϑ. The orthogonality condition (2.21) supplies the constitutive equation

$$\vartheta_{,i} = -\gamma(q_j)[(\partial\gamma/\partial q_k)q_k]^{-1}(\partial\gamma/\partial q_i). \tag{3.3}$$

If Φ is *quasi-homogeneous*, we have

$$(\partial\gamma/\partial q_k)q_k = F(\gamma) \qquad \text{and hence} \quad \vartheta_{,i} = -[\gamma(q_j)/F(\gamma)](\partial\gamma/\partial q_i). \tag{3.4}$$

If Φ is *homogeneous* of degree r, $F(\gamma)$ is equal to $r\gamma$, and the constitutive equation becomes

$$\vartheta_{,i} = -(1/r)(\partial\gamma/\partial q_i). \tag{3.5}$$

In the case of an *isotropic* matrerial, Φ has the form

$$\Phi = (1/\vartheta)\gamma(q_{(1)}), \tag{3.6}$$

where $\gamma(q_{(1)})$ is positive definite and $q_{(1)} = q_{ii}$ is the only basic invariant [see (A.1) in the Appendix] of the heat flow vector. On account of (A.6) the general constitutive equation (3.3) becomes

$$\vartheta_{,i} \approx -[\gamma(q_{(1)})/q_{(1)}]q_i. \tag{3.7}$$

In the *linear* case, the general dissipation function (3.2) reduces to

$$\Phi = (1/\vartheta)\gamma_{ij}q_iq_j, \tag{3.8}$$

where γ_{ij} is a symmetric tensor, positive definite and possibly dependent on ϑ. The dissipation function (3.8) is homogeneous of degree 2, and the orthogonality condition supplies the well-known differential equation of heat conduction in an anisotropic body,

$$\vartheta_{,i} = -\gamma_{ij}q_j. \tag{3.9}$$

Here, it is particularly easy to see (Z.15.4) that the orthogonality condition excludes gyroscopic forces, for an antimetric part of γ_{ij} would represent gyroscopic terms.

In a *linear isotropic* material we have

$$\Phi = (\gamma/\vartheta)q_{(1)}, \tag{3.10}$$

where γ is a positive scalar, possibly dependent on ϑ. The corresponding differential equation of heat conduction,

$$\vartheta_{,i} = -\gamma q_i, \tag{3.11}$$

is equivalent to Fourier's law (Z.15.4). Let us stress that even this simple case, where the vectors q_i and $\vartheta_{,i}$ have opposite drictions, *relies on orthogonality*. To be sure, one is tempted to explain the collinearity of q_i

and $\vartheta_{,i}$ in an isotropic body by considerations of material symmetry. However, a gyroscopic term in the force $-\vartheta_{,i}/\vartheta$ (easily explained, e.g., in terms of the molecular motion with respect to the rotating earth or in its magnetic field) would be compatible with the isotropy of the material, but it would modify (3.11).

IV. Elastic Solids

A. Linear Elasticity

As mentioned at the beginning of Section II,B, the elastic material may be defined by the absence of internal parameters and dissipative stresses. In the theory of elasticity it is customary to neglect thermal effects, assuming, for example, that the deformation are *isothermal.* Thus, not only the dissipative stresses but also the dissipative force (3.1) corresponding to heat flow are zero. It follows that the dissipation function is naught and the deformation is reversible. The density may be treated as constant (Z.5.3). The independent state variables are the strains ε_{ij}; the velocities are the deformation rates $d_{ij} = \dot{\varepsilon}_{ij}$, and the corresponding forces per unit volume are the stresses σ_{ij}. The most general case is governed by the functions

$$\Phi = 0, \qquad \Psi = \rho\psi(\varepsilon_{ij}), \tag{4.1}$$

where ψ is positive definite. According to (2.12) the only constitutive equation is

$$\sigma_{ij} = \partial\Psi/\partial\varepsilon_{ij}. \tag{4.2}$$

If the material is *isotropic,* Ψ assumes the form

$$\Psi(\varepsilon_{(1)}, \varepsilon_{(2)}, \varepsilon_{(3)}), \tag{4.3}$$

where $\varepsilon_{(1)} = \varepsilon_{ii}$, $\varepsilon_{(2)} = \varepsilon_{ij}\varepsilon_{ji}$, $\varepsilon_{(3)} = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}$ are the basic invariants (A.2) of the strain tensor. On account of (A.6) the general constitutive equation (4.2) becomes

$$\sigma_{ij} = \frac{\partial\Psi}{\partial\varepsilon_{(1)}}\delta_{ij} + 2\frac{\partial\Psi}{\partial\varepsilon_{(2)}}\varepsilon_{ij} + 3\frac{\partial\Psi}{\partial\varepsilon_{(3)}}\varepsilon_{ik}\varepsilon_{kj}. \tag{4.4}$$

In the *linear* case, the general expression (4.1) for the free energy reduces to

$$\Psi = \tfrac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}, \tag{4.5}$$

where c_{ijkl} is a postivie definite tensor of order 4 obeying the symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{4.6}$$

Here, (4.2) yields the well-known stress–strain relations

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}. \tag{4.7}$$

In a *linear isotropic* material we have

$$\Psi = (\lambda/2)\varepsilon_{(1)}^{2} + \mu\varepsilon_{(2)}, \tag{4.8}$$

where λ and μ are Lamé's constants, satisfying the inequalities

$$\mu > 0, \qquad 3\lambda + 2\mu > 0. \tag{4.9}$$

The stresses are now given by the generalization

$$\sigma_{ij} = \lambda\varepsilon_{(1)}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{4.10}$$

of Hooke's law.

The assumption of isothermal deformations is an idealization, useful if they are sufficiently slow. If they are fast, as, for example, in elastic vibrations, it is more realistic to treat them as *adiabatic*. It can be shown [see, e.g., (Z.5.3)] that in this case equations (4.1) through (4.10) remain valid with modified values of c_{ijkl} and λ, provided that ψ is interpreted as the specific internal energy, written as a function of ε_{ij} and s. Since, in the adiabatic case, entropy supply is zero as well as entropy production, s remains constant, and ψ appears as a function of ε_{ij} alone.

B. THERMOELASTICITY

If the deformation of an elastic body is neither isothermal nor adiabatic, the strain tensor has to be supplemented by the additional independent state variable ϑ. There also appears an additional velocity q_i with the corresponding force (3.1) per unit volume. Both leading functions $\Psi(\varepsilon_{ij}, \vartheta)$ and $\Phi(q_i)$ are now generally different from zero; the last one may also depend on ε_{ij} and ϑ. The stress is still quasi-conservative and given by (4.2). Equation (2.13) may be used to obtain the entropy per unit volume,

$$S = -\partial\Psi/\partial\vartheta; \tag{4.11}$$

(2.3) supplies the internal energy,

$$U = \Psi + \vartheta S, \tag{4.12}$$

and heat conduction is governed by (3.3), where $\gamma(q_j)$ is possibly also a function of ε_{ij} and ϑ.

If the material is *isotropic*, Ψ has the form

$$\Psi(\varepsilon_{(1)}, \varepsilon_{(2)}, \varepsilon_{(3)}, \vartheta), \tag{4.13}$$

and Φ is given by (3.6), where $\gamma(q_{(1)})$ may also depend on the arguments of Ψ. The stresses follow from (4.4), and heat conduction obeys (3.7).

In the *linear* case, it is convenient to start from a reference state where $\sigma_{ij} = 0$, $\vartheta = \vartheta_0$, and to measure ε_{ij} from this state. The free energy may be obtained by expanding Ψ with respect to the small quantities ε_{ij}, $\vartheta - \vartheta_0$ and by truncating the expansion after the second-order terms. We thus obtain

$$\Psi = \Psi_0 - S_0(\vartheta - \vartheta_0) + \tfrac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \kappa_{ij}\varepsilon_{ij}(\vartheta - \vartheta_0) - \frac{\rho c}{2\vartheta_0}(\vartheta - \vartheta_0)^2, \quad (4.14)$$

where the linear term in ε_{ij} alone is omitted for a reason that will be presently explained and where (Z.7.4) the constants Ψ_0, S_0, c_{ijkl}, κ_{ij}, c are in turn the free energy and the entropy in the reference state, the fourth-order tensor of elasticity constants introduced in Section IV,A, the tensor governing thermal stresses, and the specific heat capacity in the reference state. The dissipation function is given by (3.8). The stresses (4.2) become

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - \kappa_{ij}(\vartheta - \vartheta_0). \quad (4.15)$$

Since (4.14) contains no linear term in ε_{ij} alone, σ_{ij} is zero as required in the reference state, and the two terms on the right of (4.15) represent the stresses due to deformation and to temperature increase, respectively. From (4.11) we obtain

$$S = S_0 + \kappa_{ij}\varepsilon_{ij} + (\rho c / \vartheta_0)(\vartheta - \vartheta_0), \quad (4.16)$$

and heat conduction obeys (3.9).

In a *linear isotropic* material (4.14) has to be replaced by

$$\Psi = \Psi_0 - S_0(\vartheta - \vartheta_0) + \frac{\lambda}{2}\varepsilon_{(1)}^2 + \mu\varepsilon_{(2)} - (3\lambda + 2\mu)\kappa\varepsilon_{(1)}(\vartheta - \vartheta_0)$$

$$- \frac{\rho c}{2\vartheta_0}(\vartheta - \vartheta_0)^2, \quad (4.17)$$

where κ is the coefficient of thermal expansion and (3.8) is to be replaced by (3.10). The stresses (4.15) become

$$\sigma_{ij} = [\lambda\varepsilon_{(1)} - (3\lambda + 2\mu)\kappa(\vartheta - \vartheta_0)]\delta_{ij} + 2\mu\varepsilon_{ij}, \quad (4.18)$$

the entropy (4.16) assumes the form

$$S = S_0 + (3\lambda + 2\mu)\kappa\varepsilon_{(1)} + \frac{\rho c}{\vartheta_0}(\vartheta - \vartheta_0), \quad (4.19)$$

and heat conduction is governed by Fourier's law (3.11).

In the special case where the deformation is *isothermal*, $\vartheta = \vartheta_0$, and (4.15), (4.18) reduce as expected to (4.7) and (4.10), respectively. If the deformation is *adiabatic*, $S = S_0$, and elimination of $\vartheta - \vartheta_0$ by means of (4.16) or (4.19) yields

$$\sigma_{ij} = [c_{ijkl} + (\vartheta_0 / \rho c)\kappa_{ij}\kappa_{kl}]\varepsilon_{kl} \quad (4.20)$$

in place of (4.7) and

$$\sigma_{ij} = [\lambda + (3\lambda + 2\mu)^2(\kappa^2\vartheta_0/\rho c)]\varepsilon_{(1)}\delta_{ij} + 2\mu\varepsilon_{ij} \qquad (4.21)$$

in place of (4.10). The stress–strain relations are thus essentially the same as in the isothermal case, but the coefficients are modified as mentioned at the end of Section IV,A. For most elastic solids, the correction is of the order of a few percent (steel 2%, aluminum 5%).

To shed some more light on Eq. (4.21), let us note that, on account of (4.17) and (4.19), the internal energy (4.12) of the linear isotropic material is

$$U = U_0 + \frac{\lambda}{2}\varepsilon_{(1)}^2 + \mu\varepsilon_{(2)} + (3\lambda + 2\mu)\kappa\vartheta_0\varepsilon_{(1)} + \frac{\rho c}{2\vartheta_0}(\vartheta^2 - \vartheta_0^2), \quad (4.22)$$

where U_0 is its value in the reference state. With (4.22) and (4.21) the first fundamental law (2.4), written in the form

$$\dot{U} = \sigma_{ij}\dot{\varepsilon}_{ij} - q_{j,j}, \qquad (4.23)$$

yields

$$q_{j,j} = -(3\lambda + 2\mu)\kappa\vartheta\dot{\varepsilon}_{(1)} - (\rho c/\vartheta_0)\vartheta\dot{\vartheta}. \qquad (4.24)$$

Besides, it follows from (4.18) that

$$\sigma_{(1)} = (3\lambda + 2\mu)[\varepsilon_{(1)} - 3\kappa(\vartheta - \vartheta_0)]. \qquad (4.25)$$

The specific heat capacity of the material [see, e.g., (Z.8.1)] is defined as the ratio $-q_{j,j}/\rho\dot{\vartheta}$. With (4.24) it becomes

$$-q_{j,j}/\rho\dot{\vartheta} = (c/\vartheta_0)\vartheta + (3\lambda + 2\mu)(\kappa\vartheta/\rho)(\dot{\varepsilon}_{(1)}/\dot{\vartheta}). \qquad (4.26)$$

If the volume of the element is constant, $\dot{\varepsilon}_{(1)} = 0$, and (4.26) supplies the specific heat

$$c_v = (c/\vartheta_0)\vartheta. \qquad (4.27)$$

If, on the other hand, the pressure is constant, we have $\dot{\sigma}_{(1)} = 0$, hence, on account of (4.25), $\dot{\varepsilon}_{(1)} = 3\kappa\dot{\vartheta}$ and thus the specific heat

$$c_p = [(c/\vartheta_0) + 3(3\lambda + 2\mu)(\kappa^2/\rho)]\vartheta = c_v[1 + 3(3\lambda + 2\mu)(\kappa^2\vartheta_0/\rho c)]. \quad (4.28)$$

Equation (4.21), valid in the adiabatic case, therefore becomes

$$\sigma_{ij} = \{\lambda + (\lambda + \tfrac{2}{3}\mu)[(c_p/c_v) - 1]\}\varepsilon_{(1)}\delta_{ij} + 2\mu\varepsilon_{ij}. \qquad (4.29)$$

The factor c_p/c_v recalls acoustics, where it appears in the velocity of propagation of small disturbances. In fact, the velocities of irrotational and equivoluminal waves in an isotropic elastic solid [see, e.g., Kolsky (1963)] would be

$$c_1 = [(\lambda + 2\mu)/\rho]^{1/2} \quad \text{and} \quad c_2 = (\mu/\rho)^{1/2}, \qquad (4.30)$$

respectively, provided that the process was isothermal. In the more realistic adiabatic case λ has to be replaced by the bracket in (4.29). Thus, c_2 remains unchanged and c_1 becomes

$$c_1 = \{[\lambda + 2\mu + (\lambda + 2\mu/3)(c_p/c_v - 1)]/\rho\}^{1/2}. \tag{4.31}$$

The inviscid liquid, to be treated in Section V,B, is obtained from the elastic solid by setting $\mu = 0$. The only possible waves are irrotational, and their velocity (4.31) assumes the well-known value

$$c_1 = [(\lambda/\rho)(c_p/c_v)]^{1/2}. \tag{4.32}$$

V. Fluids

A. GASES

The fluids to be treated in this section will be assumed to be isotropic. They may be defined by the absence of internal parameters and the condition that the quasi-conservative stress be isotropic. Absence of internal parameters and isotropy of the material suggest a free energy of the type (4.13). The quasi-conservative stress is thus given by (4.4), and since it is to be isotropic, Ψ is indpendent of $\varepsilon_{(2)}$ and $\varepsilon_{(3)}$. It follows that $\varepsilon_{(1)}$ and ϑ might be used as independent state variables. In a gas, however, all deformations are possibly large; hence $\varepsilon_{(1)}$ is not a convenient variable. It is customary to replace it by the volume v of a mole. If m is the molecular mass, the specific volume is v/m, and since the specific power of the pressure p is

$$l = -(p/m)\dot{v}, \tag{5.1}$$

the quasi-conservative force associated with v is $-p/m$.

In the case of an *inviscid* gas, the dissipation function is independent of the deformation rates and hence of the form (3.6). The specific leading functions are therefore

$$\psi(v, \vartheta), \qquad \varphi = (1/\rho\vartheta)\gamma(q_{(1)}), \tag{5.2}$$

where the function γ may also depend on v and ϑ. In analogy to (2.12) and according to (2.13), we have

$$-p/m = \partial\psi/\partial v, \qquad s = -\partial\psi/\partial\vartheta. \tag{5.3}$$

The stress tensor is

$$\sigma_{ij} = -p\delta_{ij}, \tag{5.4}$$

and heat conduction is governed by (3.7).

The *ideal gas* is characterized (Z.8.1) by the molar free energy

$$m\psi = \int c_v \, d\vartheta - \vartheta \int \frac{c_v}{\vartheta} \, d\vartheta - R\vartheta \ln \frac{v}{m}, \qquad (5.5)$$

where R is the gas constant and $c_v(\vartheta)$ the molar heat capacity (in contradistinction to the heat capacities in Section IV,B, which were referred to the unit of mass). Applying the first equation (5.3) to ψ, we obtain the well-known equation of state

$$p = R\vartheta / v. \qquad (5.6)$$

Heat conduction follows from the second equation (5.2) and obeys (3.7). The second equation (5.3) supplies the entropy per mole,

$$ms = \int \frac{c_v}{\vartheta} \, d\vartheta + R \ln \frac{v}{m}, \qquad (5.7)$$

and (2.3) yields the molar internal energy,

$$mu = \int c_v \, d\vartheta, \qquad (5.8)$$

dependent on temperature alone.

In the case of a *real gas*, the molar free energy given in (5.5) has to be replaced by

$$m\psi = \int c_v \, d\vartheta - \vartheta \int \frac{c_v}{\vartheta} \, d\vartheta - R\vartheta \ln \frac{v-b}{m} - \frac{a}{v}, \qquad (5.9)$$

where a and b are van der Waal's constants, measuring, respectively, the cohesion between the molecules and their proper volume. In place of (5.6) we now obtain the equation of state of van der Waals,

$$p = R\vartheta / (v-b) - a/v^2 \qquad (5.10)$$

[compare, e.g., Hatsopoulos and Keenan (1965)]. The molar entropy becomes

$$ms = \int \frac{c_v}{\vartheta} \, d\vartheta + R \ln \frac{v-b}{m}, \qquad (5.11)$$

and the internal energy

$$mu = \int c_v \, d\vartheta - \frac{a}{v} \qquad (5.12)$$

is now a function of ϑ *and* v.

For *viscous* gases, the quasi-conservative stress (5.4) must be supplemented by a dissipative stress tensor, to be calculated as in the next section. For linear viscosity the total stress is given by (5.28) or (5.30), where d_{ij} is the deformation rate and p follows from (5.6) or (5.10).

In thermodynamics, inviscid gases have long played the preeminent role. As a consequence, the significance of the equation of state is often overrated, and this raises probems as soon as inelastic materials are to be considered. Once the fundamental significance of the governing functions ψ and φ is recognized, these problems disappear and the equation of state becomes a mere accessory. The inviscid gases just treated are defined by their free energies, (5.5) and (5.9), respectively; their equations of state, (5.6) and (5.10), follow from ψ by means of the first relation (5.3). For viscous gases, (5.6) or (5.10) yield merely the quasi-conservative part $-p\delta_{ij}$ of the stress, and Eqs. (5.28) or (5.30), dependent also on φ and supplying the total stress, are by no means equations of state.

B. LIQUIDS

In contrast to gases, the volume changes of liquids are small. Thus, the molar volume may be replaced as an independent variable by $\varepsilon_{(1)}$, and ρ may be treated as a constant. The quasi-conservative stress is still isotropic and hence of the form $-p\delta_{ij}$. Its specific power (2.1) is

$$l = (1/\rho)\sigma_{ij}\dot{\varepsilon}_{ij} = -(p/\rho)\delta_{ij}\dot{\varepsilon}_{ij} = -(p/\rho)\dot{\varepsilon}_{(1)}; \qquad (5.13)$$

the only quasi-conservative force is thus $-p/\rho$, where p is the hydrostatic pressure.

In an *inviscid* liquid, the leading functions are

$$\Psi(\varepsilon_{(1)}, \vartheta), \qquad \Phi = (1/\vartheta)\gamma(q_{(1)}), \qquad (5.14)$$

where γ may also depend on $\varepsilon_{(1)}$ and ϑ. The stress, following from (4.2) and (A.6), is

$$\sigma_{ij} = -p\delta_{ij}, \qquad p = -\partial\Psi/\partial\varepsilon_{(1)}. \qquad (5.15)$$

Heat conduction, determined by Φ, obeys (3.7), and the entropy might be obtained from (4.11).

In the *linear* case, the free energy follows, as observed at the end of Section IV,B, from (4.17) by setting $\mu = 0$. We thus have

$$\Psi = \Psi_0 - S_0(\vartheta - \vartheta_0) + \frac{\lambda}{2}\varepsilon_{(1)}^2 - 3\lambda\kappa\varepsilon_{(1)}(\vartheta - \vartheta_0) - \frac{\rho c}{2\vartheta_0}(\vartheta - \vartheta_0)^2. \quad (5.16)$$

The dissipation function is given by (3.10). The stress (4.18) becomes

$$\sigma_{ij} = -p\delta_{ij}, \qquad p = \lambda[3\kappa(\vartheta - \vartheta_0) - \varepsilon_{(1)}], \qquad (5.17)$$

and heat conduction is governed by Fourier's law (3.11).

The *incompressible* liquid is obtained by letting $\varepsilon_{(1)} \to 0$ and $\lambda \to \infty$. Together with Ψ, the hydrostatic pressure becomes indeterminate.

In a *viscous* liquid the velocity q_i has to be supplemented by the deformation rate d_{ij}, where $d_{(1)} = \dot{\varepsilon}_{(1)}$. Apart from the basic invariants $d_{(1)}$, $d_{(2)}$, $d_{(3)}$, and $q_{(1)}$, defined by (A.2) and (A.1), one generally also needs the mixed invariants (A.3), $m_{(1)} = q_i d_{ij} q_j$ and $m_{(2)} = q_i d_{ij} d_{jk} q_k$. The free energy per unit volume is still given by the first expression (5.14); the dissipation function (Z.15.4), however, becomes

$$\Phi(d_{(1)}, d_{(2)}, d_{(3)}, q_{(1)}, m_{(1)}, m_{(2)}). \tag{5.18}$$

Two cases have to be distinguished, discussed already in Section II,C.

If deformation and heat flow are *coupled*, the only way of establishing constitutive equations is to apply the orthogonality condition in the nine-dimensional space of the velocities d_{ij} and q_i. The dissipative forces are given by (2.30) and (2.31), where the terms with $\dot{\alpha}_{ij}$ are to be dropped since internal parameters are absent. The constitutive equations, obtained by means of (2.32) and Section X,B,2, are

$$\sigma_{ij} = -p\delta_{ij} + \nu \left[\frac{\partial \Phi}{\partial d_{(1)}} \delta_{ij} + 2\frac{\partial \Phi}{\partial d_{(2)}} d_{ij} + 3\frac{\partial \Phi}{\partial d_{(3)}} d_{ik} d_{kj} \right.$$
$$\left. + \frac{\partial \Phi}{\partial m_{(1)}} q_i q_j + \frac{\partial \Phi}{\partial m_{(2)}} (q_i d_{jk} + q_j d_{ik}) q_k \right], \tag{5.19}$$

$$\vartheta_{,i} = -2\nu\vartheta \left(\frac{\partial \Phi}{\partial q_{(1)}} q_i + \frac{\partial \Phi}{\partial m_{(1)}} d_{ij} q_j + \frac{\partial \Phi}{\partial m_{(2)}} d_{ij} d_{jk} q_k \right),$$

where p is given by the second equation (5.15) and

$$\nu = \Phi \left(\frac{\partial \Phi}{\partial d_{(1)}} d_{(1)} + 2\frac{\partial \Phi}{\partial d_{(2)}} d_{(2)} + 3\frac{\partial \Phi}{\partial d_{(3)}} d_{(3)} \right.$$
$$\left. + 2\frac{\partial \Phi}{\partial q_{(1)}} q_{(1)} + 3\frac{\partial \Phi}{\partial m_{(1)}} m_{(1)} + 4\frac{\partial \Phi}{\partial m_{(2)}} m_{(2)} \right)^{-1}. \tag{5.20}$$

If, on the other hand, deformation and heat flow are *independent*, the orthogonality condition has to be applied to the two processes separately. The dissipative forces are given by (2.27) and (2.28), where the terms with $\dot{\alpha}_{ij}$ are again to be dropped. The dissipation function, similar to (2.29), has the form

$$\Phi = \Phi_1(d_{(1)}, d_{(2)}, d_{(3)}) + (1/\vartheta)\gamma(q_{(1)}). \tag{5.21}$$

The stresses become

$$\sigma_{ij} = -p\delta_{ij} + \nu \left(\frac{\partial \Phi_1}{\partial d_{(1)}} \delta_{ij} + 2\frac{\partial \Phi_1}{\partial d_{(2)}} d_{ij} + 3\frac{\partial \Phi_1}{\partial d_{(3)}} d_{ik} d_{kj} \right), \tag{5.22}$$

where p is again given by the second equation (5.15) and

$$\nu = \Phi_1 \left(\frac{\partial \Phi_1}{\partial d_{(1)}} d_{(1)} + 2 \frac{\partial \Phi_1}{\partial d_{(2)}} d_{(2)} + 3 \frac{\partial \Phi_1}{\partial d_{(3)}} d_{(3)} \right)^{-1}. \qquad (5.23)$$

For heat conduction one obtains (3.7).

It has been observed in Section II,C that there is no way to decide at present between the two cases just presented. Let us add a few results for the case where the two processes are independent.

If the liquid last considered is *free of bulk viscosity*, Φ has the form

$$\Phi = \Phi_1(d'_{(2)}, d'_{(3)}) + (1/\vartheta)\gamma(q_{(1)}), \qquad (5.24)$$

where the primes designate the deviatoric part of the deformation rate. On account of (A.12) the corresponding stresses are

$$\sigma_{ij} = -p\delta_{ij} + \nu \left[2\frac{\partial \Phi_1}{\partial d'_{(2)}} d'_{ij} + 3\frac{\partial \Phi_1}{\partial d'_{(3)}} \left(d'_{ik} d'_{kj} - \frac{1}{3} d'_{(2)} \delta_{ij} \right) \right], \qquad (5.25)$$

where, according to (A.13),

$$\nu = \Phi_1 \left(2\frac{\partial \Phi_1}{\partial d'_{(2)}} d'_{(2)} + 3\frac{\partial \Phi_1}{\partial d'_{(3)}} d'_{(3)} \right)^{-1}. \qquad (5.26)$$

The hydrostatic stress is still given by the second equation (5.15) and heat conduction by (3.7).

The *Newtonian* liquid may be obtained as the linear case of (5.19), corresponding to quadratic functions Ψ and Φ, that is, to (5.16) and

$$\Phi = \lambda' d_{(1)}^2 + 2\mu' d_{(2)} + (\gamma/\vartheta)q_{(1)}, \qquad (5.27)$$

where λ' and μ', possibly dependent on $\varepsilon_{(1)}$ and ϑ, determine the viscosity in a similar manner as Lamé's constants determine the elasticity of a linear isotropic solid. Since the dissipation function (5.27) is of the type (5.21), deformation and heat flow are automatically independent. According to (5.23) $\nu = \frac{1}{2}$, and (5.22) reduces to

$$\sigma_{ij} = (-p + \lambda' d_{(1)})\delta_{ij} + 2\mu' d_{ij}, \qquad (5.28)$$

where p is given by the second equation (5.17). Heat conduction is governed by (3.11).

If the *Newtonian* liquid is *free of bulk viscosity*, (5.27) has to be replaced by

$$\Phi = 2\mu' d'_{(2)} + (\gamma/\vartheta)q_{(1)}, \qquad (5.29)$$

and (5.28) reduces to

$$\sigma_{ij} = -p\delta_{ij} + 2\mu' d'_{ij}, \qquad (5.30)$$

with the second equation (5.17) and (3.11) still valid.

Incompressible liquids will be treated in the next section.

C. INCOMPRESSIBILITY

There are two ways of dealing with incompressible materials. To discuss them, let us assume isotropy, absence of internal parameters, and indpendence of deformation and heat flow. Heat conduction can then be treated separately by the methods of Section III.

The *first approach* disregards incompressibility, that is, the conditions $\varepsilon_{(1)} = 0$, $d_{(1)} = 0$, as long as possible and introduces them only in the final results. Starting from the leading functions

$$\Psi(\varepsilon_{(1)}, \varepsilon_{(2)}, \varepsilon_{(3)}, \vartheta), \qquad \Phi(d_{(1)}, d_{(2)}, d_{(3)}) \tag{5.31}$$

(where Φ might also depend on the arguments of Ψ) and applying the methods expounded in Section II,C, together with (A.6), one obtains

$$\sigma_{ij}^{(q)} = \frac{\partial \Psi}{\partial \varepsilon_{ij}} = \frac{\partial \Psi}{\partial \varepsilon_{(1)}} \delta_{ij} + 2 \frac{\partial \Psi}{\partial \varepsilon_{(2)}} \varepsilon_{ij} + 3 \frac{\partial \Psi}{\partial \varepsilon_{(3)}} \varepsilon_{ik} \varepsilon_{kj} \tag{5.32}$$

and

$$\sigma_{ij}^{(d)} = \nu \frac{\partial \Phi}{\partial d_{ij}} = \nu \left(\frac{\partial \Phi}{\partial d_{(1)}} \delta_{ij} + 2 \frac{\partial \Phi}{\partial d_{(2)}} d_{ij} + 3 \frac{\partial \Phi}{\partial d_{(3)}} d_{ik} d_{kj} \right), \tag{5.33}$$

where

$$\nu = \Phi \left(\frac{\partial \Phi}{\partial d_{(1)}} d_{(1)} + 2 \frac{\partial \Phi}{\partial d_{(2)}} d_{(2)} + 3 \frac{\partial \Phi}{\partial d_{(3)}} d_{(3)} \right)^{-1}. \tag{5.34}$$

With $\varepsilon_{(1)} \to 0$ the derivative $\partial \Psi / \partial \varepsilon_{(1)}$ becomes indeterminate. The first term on the right of (5.32) may be written $-p\delta_{ij}$ and represents a hydrostatic pressure. Since it is indeterminate, subtraction of a term $(\partial \Psi / \partial \varepsilon_{(3)}) \varepsilon_{(2)} \delta_{ij}$ does not affect the result. The modified equation (5.32) reads

$$\sigma_{ij}^{(q)} = -p\delta_{ij} + 2 \frac{\partial \Psi}{\partial \varepsilon_{(2)}} \varepsilon_{ij} + 3 \frac{\partial \Psi}{\partial \varepsilon_{(3)}} \left(\varepsilon_{ik} \varepsilon_{kj} - \frac{1}{3} \varepsilon_{(2)} \delta_{ij} \right), \tag{5.35}$$

where the terms containing the derivatives of Ψ are deviators. A similar reasoning, applied to (5.33), yields

$$\sigma_{ij}^{(d)} = \nu \left[2 \frac{\partial \Phi}{\partial d_{(2)}} d_{ij} + 3 \frac{\partial \Phi}{\partial d_{(3)}} \left(d_{ik} d_{kj} - \frac{1}{3} d_{(2)} \delta_{ij} \right) \right] \tag{5.36}$$

and

$$\nu = \Phi \left(2 \frac{\partial \Phi}{\partial d_{(2)}} d_{(2)} + 3 \frac{\partial \Phi}{\partial d_{(3)}} d_{(3)} \right)^{-1}, \tag{5.37}$$

where an indeterminate hydrostatic term has been dropped since the principle of absent dissipative forces, established in (Z.14.3) in connection with orthogonality, requires that dissipative forces whose corresponding velocities do not appear in Φ are zero. Incidentally, (5.36) might also be obtained as the deviatoric part of (5.25) for $d_{(1)} \to 0$.

The *second approach* recognizes the incompressibility conditions from the beginning, starting from the leading functions

$$\Psi(\varepsilon_{(2)}, \varepsilon_{(3)}, \vartheta), \qquad \Phi(d_{(2)}, d_{(3)}) \tag{5.38}$$

and introducing $\varepsilon_{(1)} = 0$, $d_{(1)} = 0$ as side conditions in the differentiations. Thus,

$$\sigma_{ij}^{(q)} = (\partial/\partial\varepsilon_{ij})(\Psi + \gamma'\varepsilon_{(1)}), \tag{5.39}$$

where γ' is a Lagrangean multiplier. By means of (A.6) we obtain

$$\sigma_{ij}^{(q)} = 2(\partial\Psi/\partial\varepsilon_{(2)})\varepsilon_{ij} + 3(\partial\Psi/\partial\varepsilon_{(3)})\varepsilon_{ik}\varepsilon_{kj} + \gamma'\delta_{ij}. \tag{5.40}$$

The hydrostatic pressure becomes

$$p = -\tfrac{1}{3}\sigma_{ii}^{(q)} = -[\gamma' + (\partial\Psi/\partial\varepsilon_{(3)})\varepsilon_{(2)}]. \tag{5.41}$$

Solving this equation for γ' and inserting the result in (5.40), we obtain (5.35). In a similar manner, the modified orthogonality condition

$$\sigma_{ij}^{(d)} = \nu(\partial/\partial d_{ij})(\Phi + \gamma''d_{(1)}) \tag{5.42}$$

yields

$$\sigma_{ij}^{(d)} = \nu[2(\partial\Phi/\partial d_{(2)})d_{ij} + 3(\partial\Phi/\partial d_{(3)})d_{ik}d_{kj} + \gamma''\delta_{ij}], \tag{5.43}$$

and the principle of absent dissipative forces, requiring $\sigma_{ij}^{(d)}$ to be a deviator, leads back to (5.36) and (5.37).

If the material just considered is a *liquid*, Ψ is independent of $\varepsilon_{(2)}$ and $\varepsilon_{(3)}$. It follows from (5.35), (5.36), and (2.32) that

$$\sigma_{ij} = -p\delta_{ij} + \nu\left[2\frac{\partial\Phi}{\partial d_{(2)}}d_{ij} + 3\frac{\partial\Phi}{\partial d_{(3)}}\left(d_{ik}d_{kj} - \frac{1}{3}d_{(2)}\delta_{ij}\right)\right], \tag{5.44}$$

where p is indeterminate and ν is given by (5.37). Equation (5.44) represents a special liquid of the Reiner (1945)–Rivlin (1948) type, characterized by coefficients of d_{ij} and of the parenthesis that are coupled by the dissipation function. More general *Reiner–Rivlin liquids*, with independent coefficients, do not satisfy the orthogonality condition. As far as we know, there is no evidence for their existence.

The often-used constitutive equation

$$\sigma_{ij} = -p\delta_{ij} + 2\mu'(d_{(2)}, d_{(3)})d_{ij} \tag{5.45}$$

defines the so-called *quasi-linear* liquid (Z.9.4). If (5.45) is to be a special case of (5.44), the coefficient of the parenthesis in (5.44) must be zero. The orthogonality condition thus requires that Φ and hence μ' depend on $d_{(2)}$ alone.

The incompressible *Newtonian* liquid is characterized by a quadratic dissipation function, that is, by

$$\Phi = 2\mu'd_{(2)}. \tag{5.46}$$

Here, (5.44) reduces to

$$\sigma_{ij} = -p\delta_{ij} + 2\mu'd_{ij}, \tag{5.47}$$

where p is still indeterminate.

Quasi-linear and Newtonian fluids are examples of materials whose dissipation functions depend on the second basic invariant $d_{(2)}$ alone. We will encounter more materials of this type and note that here (5.36) and (5.37) reduce to the simple equation

$$\sigma_{ij}^{(d)} = [\Phi(d_{(2)})/d_{(2)}]\,d_{ij}. \tag{5.48}$$

The process of specialization carried through in this section can be inverted: the incompressible Newtonian liquid can be generalized starting from the dissipation function

$$\Phi = 2\mu'd_{(2)} + (\gamma/\vartheta)q_{(1)} \tag{5.49}$$

and adding terms of successively higher degree in d_{ij} and q_i, which are expressible in the basic invariants. This has been done elsewhere for independent (Z.16.1) and for coupled (Z.16.3) processes.

VI. Plasticity

A. Rigid, Perfectly Plastic Materials

In this and the remaining sections we will assume that deformation and heat flow are independent. The dissipation function then consists of two parts, dependent, respectively, on deformation and heat flow. The second term may be dropped provided that one uses the appropriate results of Section III (with γ possibly dependent on deformation and temperature) in order to obtain heat conduction.

It has been shown by Houlsby (1979, 1980, 1981a,b) that the treatment of plastic materials such as ductile metals and even of soils may be used on appropriate function Ψ and Φ.

The dissipation function of an arbitrary material can be represented geometrically by means of dissipation surfaces $\Phi = \text{const}$ in the space R of the principal deformation rates $d_1, \ldots,$ or by surfaces $\Phi' = \text{const}$ in the space R' of the principal stresses $\sigma_1, \ldots$ If the mapping between the two spaces is one–one, both families of surfaces have the properties discussed for the φ surfaces in connection with the orthogonality condition in Section II,B.

In the case of an incompressible liquid, $d_{(1)} = 0$. Thus, Φ is only defined in the deviatoric plane E (see Section X,B,4) of the space R. It may be represented by curves $\Phi = \text{const}$ in E. Since $\sigma_{ij}^{(d)}$ is a deviator, (5.44) maps

these curves onto the deviatoric plane E' of the space R', and addition of an arbitrary hydrostatic stress supplies surfaces $\Phi' = \text{const}$ in the form of cylinders with axes perpendicular to E'.

The rigid, perfectly plastic material (Z.17.2) may be defined as a special case of the incompressible liquid, characterized by a dissipation function (5.38) that is homogeneous of the first degree in d_{ij} (and independent of the state variables). Here, (5.37) yields $\nu = 1$, and (5.44) supplies the stress deviator

$$\sigma'_{ij} = 2(\partial\Phi/\partial d_{(2)})d_{ij} + 3(\partial\Phi/\partial d_{(3)})(d_{ik}d_{kj} - \tfrac{1}{3}d_{(2)}\delta_{ij}), \qquad (6.1)$$

whereas the isotropic part of the stress tensor remains indeterminate.

Let us consider the values of Φ on an arbitrary ray s emanating from the origin O in the plane E. Since the dissipation function is homogeneous of the first degree, Φ increases proportional to the distance from O on any such ray. On account of (6.1) σ'_{ij} is the same for all points on s. It follows that the curves $\Phi' = \text{const}$ in E' coincide and define a yield surface in the shape of a cylinder with axis perpendicular to E'. Since Φ has been assumed to be independent of the state variables, the corollary (Section II,B) of the orthogonality condition holds. Thus, the cylinder, considered as the limiting case of a dense layer of Φ surfaces, is convex, and the vector **d** representing the strain rate lies in its outward normal in the end point of the vector $\boldsymbol{\sigma}$ but is of indeterminate magnitude since σ'_{ij}, on account of (6.1), is homogeneous of degree 0 in d_{ij}. The vector **d** thus obeys the so-called normality condition, and this is sometimes expressed by saying that it is associated with the yield condition, that is, with the equation of the yield surface. Convexity of the yield surface and the normality condition together represent (Z.10.2) what is usually called the theory of the plastic potential.

The simplest case is the *v. Mises* material, defined by a dissipation function that is independent of $d_{(3)}$ and hence of the form

$$\Phi = k(2d_{(2)})^{1/2}, \qquad (6.2)$$

where k is a scalar (possibly dependent on ϑ). The dissipation function (6.2) is constant, according to (A.14), on circles in E around the origin O and proportional to their radii. The deviatoric stress (6.1) or (5.48) becomes

$$\sigma'_{ij} = k(\tfrac{1}{2}d_{(2)})^{-1/2}d_{ij}, \qquad (6.3)$$

and it follows that

$$\sigma'_{(2)} = \sigma'_{ij}\sigma'_{ji} = \sigma'^2_1 + \cdots = 2k^2, \qquad (6.4)$$

where (A.2) has been used. This equation evidently represents the yield condition. In the deviatoric plane of space R' it is to be interpreted as a circle (Fig. 2) of radius $k\sqrt{2}$, and the entire yield surface is the corresponding circular cylinder. Equation (6.3) (written in principal values) represents the
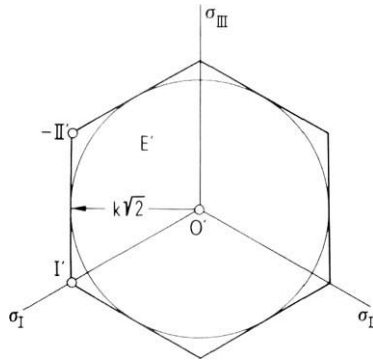
FIG. 2.   Yield loci of v. Mises and Tresca material.

normality condition, and from (6.4) we finally conclude that k is the yield stress in simple shear.

In the case of a *Tresca* material the dissipation function depends on $d_{(2)}$ and $d_{(3)}$. Since its structure is complicated, it is convenient to define it implicitly (Z.17.2). However, in the deviatoric plane it allows the simple representation

$$\Phi = \begin{cases} \pm 2kd_{\mathrm{I}} & (\mathrm{I}, -\mathrm{I}), \\ \pm 2kd_{\mathrm{II}} & (\mathrm{II}, -\mathrm{II}), \\ \pm 2kd_{\mathrm{III}} & (\mathrm{III}, -\mathrm{III}), \end{cases} \tag{6.5}$$

where the roman numerals refer to the six sectors subdividing the entire plane of Fig. 3. On account of (A.15) the function Φ is constant on regular hexagons, one of which is outlined in Fig. 3. In order to obtain the stresses we recall that, on account of (5.42), Eq. (6.1) is equivalent to the orthogonality condition

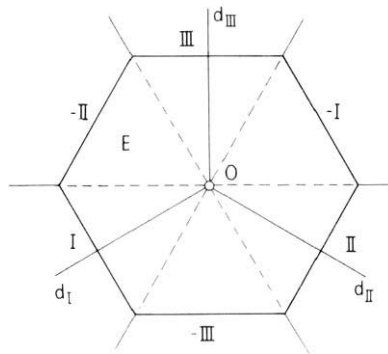$$\sigma'_{ij} = (\partial/\partial d_{ij})(\Phi + \gamma' d_{(1)}), \tag{6.6}$$

FIG. 3.   Sectors corresponding to the six definitions of Φ in (6.5).

where the Lagrangean multiplier γ' is to be determined so that the right-hand side is a deviator. In the open region I of Fig. 3, (6.6) supplies

$$\sigma_I' = \tfrac{4}{3}k, \qquad \sigma_{II}' = \sigma_{III}' = -\tfrac{2}{3}k; \tag{6.7}$$

the interior of region I is thus mapped onto a single point I' on the projected axis σ_I in Fig. 2. The other open regions in Fig. 3 yield the remaining corners of a regular hexagon with center O'. On the line dividing regions I and $-$II in Fig. 3 we have an additional side condition, $d_I + d_{II} = 0$. Inserting it with a Lagrangean multiplier γ'' in the orthogonality condition (6.6), we obtain

$$\sigma_I' = \tfrac{1}{3}(4k + \gamma''), \qquad \sigma_{II}' = -\tfrac{1}{3}(2k - \gamma''), \qquad \sigma_{III}' = -\tfrac{2}{3}(k + \gamma'') \tag{6.8}$$

in place of (6.7). Since these equations represent the straight line connecting the points I' and $-II'$ in Fig. 2, the yield locus is a regular hexagon with sides parallel to the projected axes $\sigma_I, \ldots$. If k is again the yield stress in simple shear, the Tresca hexagon, usually obtained by the condition that the maximal shearing stress be equal to k, circumscribes the v. Mises circle. The yield surface is the prism with the hexagonal cross section of Fig. 2.

Tresca yield is an example of an irregular dissipation function. Its gradient is not defined on the boundaries between the six sectors of Fig. 3, and this corresponds to the corners of the hexagons on which Φ is constant. However, any difficulties can be avoided by rounding the corners, that is, by considering them as limiting cases of smooth curves. The correspondence between the points in Fig. 3 and those of the yield locus then shows that the flow rule associated with the Tresca yield surface is satisfied.

B. Construction of the Dissipation Function

The connection with analytical mechanics pointed out in Section I and the fact (Section II,B) that orthogonality need not hold in force space show that the dissipation function Φ in velocity space deserves priority over Φ' and, in the present case, over the yield locus. Experimentally, however, it is easier to determine the yield locus than the dissipation function. It has been established, for example, that certain ductile metals are neither exactly v. Mises nor Tresca materials [see, e.g., Hill (1950)]. However, the analytical formulation of a more reliable yield condition is not easy, and the construction of the corresponding dissipation function would present another problem. We will confine ourselves, therefore, to the demonstration that the theory of the plastic potential, based on a prescribed yield condition, allows one in principle to construct the corresponding dissipation function.

Let

$$F(\sigma_{ij}') = 0 \qquad [F(0) < 0] \tag{6.9}$$

be the equation of a yield locus in the plane E', convex and star shaped

with respect to the origin, and let us forget for a moment that it might be expressed in the basic invariants $\sigma'_{(2)}$ and $\sigma'_{(3)}$. For a stress state at the yield limit the normality condition

$$d_{ij} = \nu(\partial F/\partial \sigma'_{ij}) \qquad (\nu \geqq 0) \tag{6.10}$$

describes the corresponding strain rates. Since ν is arbitrary, each point on the yield locus (6.9) is mapped onto an entire ray s emanating from the origin in E.

Let us first assume that the yield locus is strongly convex. If we let the stress point move along it in a prescribed sense, the outward normal rotates nonstop in the same sense, and so does the image s in the plane E. The value of the dissipation function in a given point of s is the scalar product of its own radius vector $\mathbf{d}$ and the radius vector $\boldsymbol{\sigma}'$ of the corresponding point on the yield locus. Since the yield locus is convex and star shaped with respect to the origin O', the product $\boldsymbol{\sigma}' \cdot \mathbf{d}$ is nonnegative. It increases proportional to the distance from O; the dissipation function obtained in this way is thus homogeneous of the first degree. Since each ray s is the image of a single vector $\boldsymbol{\sigma}'$ (even if the yield locus has corners), the dissipation function is single valued.

If the yield locus is merely weakly convex, it contains at least one straight section. The stress points lying on it correspond to a single ray s in E. However, since s is orthogonal to the straight section, the scalar product $\boldsymbol{\sigma}' \cdot \mathbf{d}$, evaluated for a given point on s, is the same for all corresponding vectors $\boldsymbol{\sigma}'$; thus, Φ is still single valued. Incidentally, that the Φ surfaces are convex and star shaped with respect to O has been shown elsewhere (Z.14.5).

C. Elastic, Perfectly Plastic Materials

If the initial response of an otherwise perfectly plastic material is elastic, we call it elastic, perfectly plastic. Its treatment requires a set of internal parameters in the form of an internal strain tensor α_{ij}. Provided that we identify it with what is usually called the plastic strain $\varepsilon_{ij}^{(p)}$, the difference $\varepsilon_{ij} - \varepsilon_{ij}^{(p)} = \varepsilon_{ij}^{(e)}$ is the elastic strain. If plastic volume changes can be excluded as in Section VI,A, $\alpha_{ij} = \varepsilon_{ij}^{(p)}$ is a deviator.

Let us assume that the elastic part of the response is linear. The governing functions then follow from (4.8) and the second expression (5.38). They are

$$\Psi = (\lambda/2)\varepsilon_{(1)}^2 + \mu(\varepsilon - \alpha)_{(2)}, \qquad \Phi(\dot{\alpha}_{(2)}, \dot{\alpha}_{(3)}), \tag{6.11}$$

where $(\varepsilon - \alpha)_{(2)}$ is the second basic invariant (A.2) of the tensor $\varepsilon_{ij} - \alpha_{ij}$, and where Φ is homogeneous of the first degree in $\dot{\alpha}_{ij}$. The external stress

is quasi-conservative and given by

$$\sigma_{ij} = (\partial/\partial\varepsilon_{ij})(\Psi + \gamma'\alpha_{(1)}) = \lambda\varepsilon_{(1)}\delta_{ij} + 2\mu(\varepsilon_{ij} - \alpha_{ij}), \qquad (6.12)$$

where γ' is a Lagrangian multiplier. Since $\varepsilon_{(1)} = \varepsilon_{(1)}^{(e)}$, (6.12) supplies the expected connection (4.10),

$$\sigma_{ij} = \lambda\varepsilon_{(1)}^{(e)}\delta_{ij} + 2\mu\varepsilon_{ij}^{(e)}, \qquad (6.13)$$

between stress and elastic strain. Equation (6.12) can be decomposed according to

$$\sigma'_{ij} = 2\mu(\varepsilon'_{ij} - \alpha_{ij}), \qquad \sigma_{(1)} = (3\lambda + 2\mu)\varepsilon_{(1)}. \qquad (6.14)$$

The quasi-conservative part of the internal stress is

$$\beta_{ij}^{(q)} = (\partial/\partial\alpha_{ij})(\Psi + \gamma'\alpha_{(1)}) = -2\mu(\varepsilon_{ij} - \alpha_{ij}) + \gamma'\delta_{ij}. \qquad (6.15)$$

From (6.15) and the first equation (6.14) it follows that

$$\beta_{ij}^{(q)'} = -\sigma'_{ij}, \qquad (6.16)$$

and by analogy with (6.1) we get the dissipative internal stress

$$\beta_{ij}^{(d)'} = 2(\partial\Phi/\partial\dot{\alpha}_{(2)})\dot{\alpha}_{ij} + 3(\partial\Phi/\partial\dot{\alpha}_{(3)})(\dot{\alpha}_{ik}\dot{\alpha}_{kj} - \tfrac{1}{3}\dot{\alpha}_{(2)}\delta_{ij}). \qquad (6.17)$$

Applying (2.35) to the deviatoric parts of β_{ij}, we finally obtain the connection

$$\sigma'_{ij} = 2(\partial\Phi/\partial\dot{\varepsilon}_{(2)}^{(p)})\dot{\varepsilon}_{ij}^{(p)} + 3(\partial\Phi/\partial\dot{\varepsilon}_{(3)}^{(p)})(\dot{\varepsilon}_{ik}^{(p)}\dot{\varepsilon}_{kj}^{(p)} - \tfrac{1}{3}\dot{\varepsilon}_{(2)}^{(p)}\delta_{ij}) \qquad (6.18)$$

between stress and plastic strain rate.

For an elastic, perfectly plastic material of the v. Mises type, (6.2) suggests the dissipative function

$$\Phi = k(2\dot{\varepsilon}_{(2)}^{(p)})^{1/2}, \qquad (6.19)$$

while the free energy is still given by the first expression (6.11),

$$\Psi = (\lambda/2)\varepsilon_{(1)}^{(e)2} + \mu\varepsilon_{(2)}^{(e)} \qquad (6.20)$$

(Houlsby, 1979). Thus, (6.18) reduces to

$$\sigma'_{ij} = k(\tfrac{1}{2}\dot{\varepsilon}_{(2)}^{(p)})^{-1/2}\dot{\varepsilon}_{ij}^{(p)}, \qquad (6.21)$$

which is analogous to (6.3). Yield surface is still the circular cylinder (6.4), and (6.21) represents the associated flow rule.

In the case of a Tresca material the dissipation function is given by (6.5), provided that we replace $d_1, \ldots,$ by $\dot{\varepsilon}_1^{(p)}, \ldots$. Yield surface is the prism with the hexagonal cross section of Fig. 2, and the plastic strain rate obeys the associated flow rule.

D. LINEAR HARDENING

The simplest case of a hardening material corresponds to the governing functions

$$\Psi = (\lambda/2)\varepsilon_{(1)}^2 + \mu(\varepsilon - \alpha)_{(2)} + \mu'\alpha_{(2)}, \qquad \Phi(\dot{\alpha}_{(2)}, \dot{\alpha}_{(3)}), \qquad (6.22)$$

where Φ is still homogeneous of the first degree in $\dot{\alpha}_{ij}$. The external stress is again quasi-conservative and given by (6.12), and the connection between stress and elastic strain is (6.13). The decomposition (6.14) is still valid, but in place of (6.15) we obtain

$$\beta_{ij}^{(q)} = (\partial/\partial\alpha_{ij})(\Psi + \gamma'\alpha_{(1)}) = -2\mu(\varepsilon_{ij} - \alpha_{ij}) + 2\mu'\alpha_{ij} + \gamma'\delta_{ij} \quad (6.23)$$

for the quasi-conservative part of the internal stress. Instead of (6.16) we now have

$$\beta_{ij}^{(q)'} = -\sigma_{ij}' + 2\mu'\alpha_{ij}, \qquad (6.24)$$

whereas the dissipative part of the internal stress is still determined by (6.17). By means of (2.35) we finally obtain the differential equation

$$\sigma_{ij}' - 2\mu'\varepsilon_{ij}^{(p)} = 2(\partial\Phi/\partial\dot{\varepsilon}_{(2)}^{(p)})\dot{\varepsilon}_{ij}^{(p)} + 3(\partial\Phi/\partial\dot{\varepsilon}_{(3)}^{(p)})(\dot{\varepsilon}_{ik}^{(p)}\dot{\varepsilon}_{kj}^{(p)} - \tfrac{1}{3}\dot{\varepsilon}_{(2)}^{(p)}\delta_{ij}) \quad (6.25)$$

connecting the stress and the plastic strain.

For a hardening material of the v. Mises type, the functions (6.22) become

$$\Psi = (\lambda/2)\varepsilon_{(1)}^{(e)2} + \mu\varepsilon_{(2)}^{(e)} + \mu'\varepsilon_{(2)}^{(p)}, \qquad \Phi = k(2\dot{\varepsilon}_{(2)}^{(p)})^{1/2} \qquad (6.26)$$

(Houlsby, 1979), and (6.25) reduces to

$$\sigma_{ij}' - 2\mu'\varepsilon_{ij}^{(p)} = k(\tfrac{1}{2}\dot{\varepsilon}_{(2)}^{(p)})^{-1/2}\dot{\varepsilon}_{ij}^{(p)}. \qquad (6.27)$$

It follows that

$$(\sigma' - 2\mu'\varepsilon^{(p)})_{(2)} = 2k^2. \qquad (6.28)$$

The yield locus (Fig. 4) in the plane E' is the circle of radius $k\sqrt{2}$ with
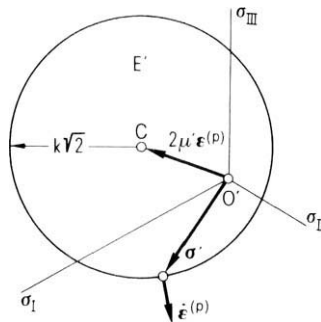

FIG. 4.   Yield locus of hardening material.

center C given by the vector $2\mu'\boldsymbol{\varepsilon}^{(p)} = 2\mu'(\varepsilon_1^{(p)},\ldots)$. Yield surface is the cylinder with the circular cross section of Fig. 4, and (6.27) represents the associated flow rule. During plastic flow, the cylinder moves in the direction of $\dot{\boldsymbol{\varepsilon}}^{(p)}$. This corresponds to Prager's hardening rule (1955), which coincides in this special case with the one of Ziegler (1959). It is remarkable that the functions Ψ and Φ (together with the condition $\varepsilon_{(1)}^{(p)} = 0$) determine not only the yield surface and the associated flow rule but also the hardening rule.

The hardening material of the Tresca type is governed by the free energy given in (6.26) and by the dissipation function

$$\Phi = \pm 2k\dot{\varepsilon}_1^{(p)} \qquad (\text{I}, -\text{I}),\ldots, \tag{6.29}$$

in connection with Fig. 3, where $d_1,\ldots,$ is to be replaced by $\dot{\varepsilon}_1^{(p)},\ldots.$ The yield locus follows from Fig. 4 provided that the circle is replaced, as in Fig. 2, by a regular hexagon. The plastic strain rate is determined by the flow rule associated with the hexagonal prism, and the hardening rule corresponding to the functions Ψ and Φ used here is the one of Prager. It is doubtful whether a pair of governing functions can be found that yields Ziegler's hardening rule.

A slightly more general type of hardening has been mentioned by Germain *et al.* (1983).

E. RATE-DEPENDENT YIELD

Experiments by Manjoine (1944) have shown that the response of certain materials is nearly elastic, perfectly plastic except that the yield stress depends on the strain rate. Materials of this type can be characterized by the free energy (6.20) and a dissipation function like (6.19) or (6.5) with $d_1,\ldots,$ replaced by $\dot{\varepsilon}_1^{(p)},\ldots,$ and with a factor k that is a function of the plastic strain rate.

A simple extension of (6.19) is the function

$$\Phi = A[1 + B(\dot{\varepsilon}_{(2)}^{(p)})^{1/n}](2\dot{\varepsilon}_{(2)}^{(p)})^{1/2}, \tag{6.30}$$

where A, B, and n are constants. Since Φ depends on $\dot{\varepsilon}_{(2)}^{(p)}$ alone, the stress deviator is given by (5.48), where d_{ij} and $\sigma_{ij}^{(d)}$ are to be replaced, respectively, by $\dot{\varepsilon}_{ij}^{(p)}$ and σ'_{ij}. We thus obtain the stress deviator

$$\sigma'_{ij} = A[1 + B(\dot{\varepsilon}_{(2)}^{(p)})^{1/n}](\tfrac{1}{2}\dot{\varepsilon}_{(2)}^{(p)})^{-1/2}\dot{\varepsilon}_{ij}^{(p)} \tag{6.31}$$

and the yield condition

$$\sigma'_{(2)} = 2A^2[1 + B(\dot{\varepsilon}_{(2)}^{(p)})^{1/n}]^2, \tag{6.32}$$

dependent on the plastic strain rate.

In the case of uniaxial stress we have

$$\sigma_1' = \tfrac{2}{3}\sigma_1. \tag{6.33}$$

Furthermore, $\dot{\varepsilon}_{II}^{(p)} = \dot{\varepsilon}_{III}^{(p)} = -\dot{\varepsilon}_{(p)}1/2$ and hence

$$\dot{\varepsilon}_{(2)}^{(p)} = \tfrac{3}{2}(\dot{\varepsilon}_1^{(p)})^2. \tag{6.34}$$

Thus, (6.31) yields

$$\sigma_I = \pm A\sqrt{3}\,[1 + B(\tfrac{3}{2})^{1/n}(\dot{\varepsilon}_1^{(p)})^{2/n}] \qquad (\dot{\varepsilon}_1^{(p)} \gtrless 0). \tag{6.35}$$

With the notations

$$A\sqrt{3} =: \sigma_0, \qquad 2/n =: 1/p, \qquad B(\tfrac{3}{2})^{1/n} =: D^{-1/p}, \tag{6.36}$$

(6.35) becomes

$$\sigma_1 = \sigma_0[1 + (\dot{\varepsilon}_1^{(p)}/D)^{1/p}] \qquad (\dot{\varepsilon}_1^{(p)} > 0) \tag{6.37}$$

or

$$\dot{\varepsilon}_1^{(p)} = D(\sigma_1/\sigma_0 - 1)^p \qquad (\sigma_1 > \sigma_0). \tag{6.38}$$

This is the relation deduced by Cowper and Symonds (1957) and Bodner and Symonds (1962) from an analysis of Manjoine's test results.

In Fig. 5, σ_1/σ_0 is plotted against $\dot{\varepsilon}_1^{(p)}/D$ for a few values of p. The parameter σ_0 is the yield stress for vanishing plastic strain rate, and D is the value of $\dot{\varepsilon}_1^{(p)}$ for which the yield stress becomes $2\sigma_0$. Comparison of
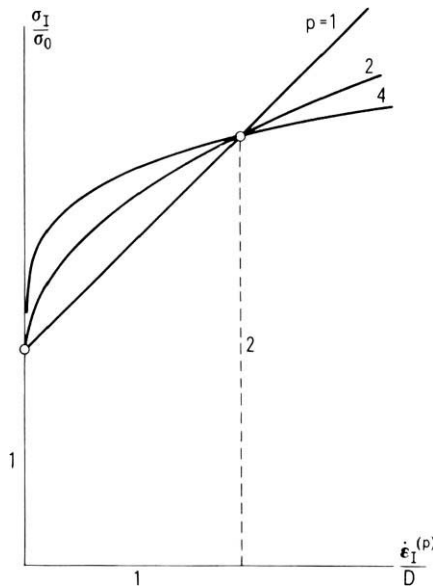

FIG. 5.   Rate-dependent yield in simple tension.

Figs. 5 and 11 shows that the material considered here might be characterized as elastic, viscoplastic.

VII. Soils

A. NONASSOCIATED FLOW

The response patterns described in the preceding section are useful models for the actual behavior of ductile metals. One is therefore tempted to assume that normality in force space and, in particular, the flow rule associated with the yield condition of a plastic material, are necessary consequences of the orthogonality condition formulated, as in Section II,B, in velocity space. However, this is not the case. Counterexamples like soil and concrete have been known for a long time. Experiments by Richmond and Spitzig (1980) have shown that certain steels and polymers subjected to high pressure contradict associated flow.

These examples do not invalidate maximal rate of entropy production. In fact, Houlsby (1979, 1981a,b) has demonstrated that, for certain materials, the orthogonality condition, as formulated in Section II,B, supplies yield conditions and flow rules that are not associated. As shown elsewhere (Z.14.3), where the orthogonality condition has been established in velocity space, it implies normality in force space only under the condition that the dissipation function depends on the velocities alone. If it also depends on the independent state variables, normality in force space is not to be expected and, as a consequence, yield conditions and flow rules need not be associated. As stated in Section I, there is no general duality between velocities and forces.

The elastic, perfectly plastic material of Section VI,C is distinguished by a dissipation function dependent on the internal strain rates alone. It hence obeys the associated flow rule. Materials like soils, on the other hand, may be characterized by governing functions of the type

$$\Psi = (\lambda/2)\varepsilon_{(1)}^2 + \mu(\varepsilon - \alpha)_{(2)}, \qquad \Phi(\dot{\alpha}_{(2)}, \dot{\alpha}_{(3)}, \varepsilon_{(1)}), \qquad (7.1)$$

together with the conditions that the internal strain $\alpha_{ij} = \varepsilon_{ij}^{(p)}$ be a deviator and that Φ be homogeneous of the first degree in the internal strain rates. The only difference with respect to (6.11) is the dependence of the dissipation function on the dilatation $\varepsilon_{(1)}$. This does not affect the reasoning leading from (6.11) to (6.18). On account of the second equation (6.14), however, the argument $\varepsilon_{(1)}$ in the dissipation function (7.1) can be replaced by $\sigma_{(1)}$. It is therefore customary to talk of *pressure-dependent yield.*

Quite a number of models have been proposed to describe the response of the materials in question. They have been collected, among others, by

Chen and Saleeb (1982). A few of them will be presently discussed, others in Section VII,B, and it will be shown that their response follows from leading functions of the type (7.1).

Case 1: The simplest case (Z.17.6) is the material with the free energy (7.1) and the particular dissipation function

$$\Phi = A[B - \sqrt{3}(\lambda + \tfrac{2}{3}\mu)\varepsilon_{(1)}]\dot{\alpha}_{(2)}^{1/2} = A[B - (\sigma_{(1)}/\sqrt{3})](\dot{\varepsilon}_{(2)}^{(p)})^{1/2}, \quad (7.2)$$

where A, B are positive constants and the inequality

$$\sqrt{3}(\lambda + \tfrac{2}{3}\mu)\varepsilon_{(1)} = \sigma_{(1)}/\sqrt{3} \leqq B \quad (7.3)$$

is to be repeated since Φ is nonnegative. From (5.48) and (7.2) we obtain the stress deviator

$$\sigma'_{ij} = A[B - \sigma_{(1)}/\sqrt{3}](\dot{\varepsilon}_{(2)}^{(p)})^{-1/2}\dot{\varepsilon}_{ij}^{(p)} \quad (7.4)$$

and the equation

$$\sigma'_{(2)} = A^2(B - \sigma_{(1)}/\sqrt{3})^2 \quad (7.5)$$

of the yield surface. In the space R' it is a circular semicone with axis g (Fig. 13 later in chapter) and the longitudinal section of Fig. 6. The vertex is determined by B, and A is the tangent of the semiaperture. The yield stress in hydrostatic tension is $\sigma_{(1)} = B\sqrt{3}$; in simple shear it is $k = AB/\sqrt{2}$ (compare Section VI,A). In uniaxial stress σ_1 we have $\sigma_{(1)} = \sigma_1$ and $\sigma'_{(2)} = 2\sigma_1^2/3$. Thus, the yield stresses in simple tension or compression are

$$\sigma_1^+ = AB\sqrt{3}/(\sqrt{2} + A), \qquad \sigma_1^- = -AB\sqrt{3}/(\sqrt{2} - A), \quad (7.6)$$

respectively. It follows that σ_1^+ always exists and is smaller than $B\sqrt{3}$, whereas σ_1^- only exists if $A < \sqrt{2}$. The perfectly plastic material follows from (7.2) by letting $B \to \infty$ and $A \to 0$ so that $AB \to k\sqrt{2}$.

The yield condition (7.5) is equivalent to the one proposed by Drucker and Prager (1952) for soils and confirmed by Richmond and Spitzig (1980) for certain steels and polymers under high pressure. Since $\varepsilon_{(1)}^{(p)} = 0$, the
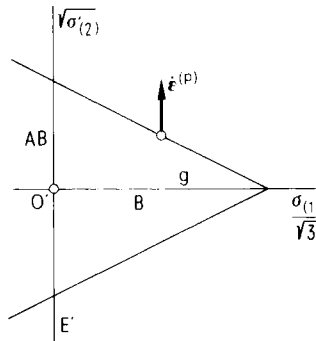

FIG. 6.   Longitudinal section of Drucker–Prager yield surface.

tensor $\dot{\varepsilon}_{ij}^{(p)}$ is a deviator. The vector $\dot{\varepsilon}^{(p)}$ in Fig. 6 is therefore parallel to the deviatoric plane E' and not normal to the yield surface as assumed by Drucker and Prager. In fact, the flow rule (7.4) is not associated with the yield condition (7.5). So far, the strongest support for normality in stress space have been Drucker's postulates (1951). They are obviously not generally valid.

At the vertex of the semicone, (7.3) holds as an equation, and (7.4) yields $\sigma_{ij}' = 0$. The corresponding vector $\dot{\varepsilon}^{(p)}$ is still parallel to E', but apart from this its direction is arbitrary.

Case 2: Certain materials, such as cohesionless soils, respond similarly to the one just treated but cannot sustain stress states with $\sigma_{(1)} > 0$. One possibility of dealing with them, proposed by Houlsby (1979), is equivalent to using (7.2) with $B = 0$. The result is a yield cone with vertex at O' and again a nonassociated flow rule.

Case 3: Another model, used extensively, is obtained by truncating the cone of Fig. 6, keeping only the portion where $\sigma_{(1)} < 0$ and closing it by a circular area in the deviatoric plane. In this plane the normality condition breaks down since it requires, together with the deviatoric character of $\dot{\varepsilon}^{(p)}$, that $\dot{\varepsilon}^{(p)} = \mathbf{0}$. However, if we start from the functions Ψ and Φ, restricting (7.2) to the domain $\sigma_{(1)} \leqq 0$ and setting $A = 0$ for $\sigma_{(1)} = 0$, Eq. (7.4) yields $\sigma_{ij}' = 0$ for $\sigma_{(1)} = 0$, and the corresponding vector $\dot{\varepsilon}^{(p)}$ represents an arbitrary deviatoric strain rate. Here, no problem with normality arises in the plane E', for the yield surface appears as a truncated semicone open at either end and complemented by a single point at the origin.

B. Various Models

A few models of soils are defined by the functions Ψ and Φ used in Section VII,A. In order to discuss additional models, we retain the expressions (7.1) for the governing functions, the assumption that Φ is homogeneous of the first degree and the postulate that the plastic strain is a deviator. We thus have a yield surface in stress space, a vector $\dot{\varepsilon}^{(p)}$ that is always parallel to the deviatoric plane E', and in general no normality in stress space.

Case 1: Let us start with the dissipation function

$$\Phi = \{A[B - (\sigma_{(1)}/\sqrt{3})]\dot{\varepsilon}_{(2)}^{(p)}\}^{1/2}, \tag{7.7}$$

where A, B are positive constants and

$$\sigma_{(1)} \leqq B\sqrt{3}. \tag{7.8}$$

From (5.48) and (7.7) we obtain the stress deviator

$$\sigma_{ij}' = \{A[B - (\sigma_{(1)}/\sqrt{3})]\}^{1/2}(\dot{\varepsilon}_{(2)}^{(p)})^{-1/2}\dot{\varepsilon}_{ij}^{(p)} \tag{7.9}$$

an⟨ he equation

$$\sigma'_{(2)} = A(B - \sigma_{(1)}/\sqrt{3}) \tag{7.10}$$

of the yield surface. In the space R' (7.10) represents a paraboloid of revolution (Fig. 7). The yield stress in hydrostatic tension is $\sigma_{(1)} = B\sqrt{3}$; in simple shear it is $k = \sqrt{AB/2}$ (compare Section VI,A).

The cohesionless case of the material just considered can be obtained either by setting $B = 0$ or by restricting (7.7) to the domain $\sigma_{(1)} \leqq 0$ and setting $A = 0$ for $\sigma_{(1)} = 0$.

Case 2: The dissipation function

$$\Phi = A\{[B - (\sigma_{(1)}/\sqrt{3})]^2 - (B - C)^2\}^{1/2}(\dot{\varepsilon}^{(p)}_{(2)})^{1/2}, \tag{7.11}$$

where $C < B$ is another positive constant beside A and B and where

$$\sigma_{(1)} \leqq C\sqrt{3} \tag{7.12}$$

is a generalization of (7.2). The corresponding stress deviator (5.48) is

$$\sigma'_{ij} = A\{[B - (\sigma_{(1)}/\sqrt{3})]^2 - (B - C)^2\}^{1/2}(\dot{\varepsilon}^{(p)}_{(2)})^{-1/2}\dot{\varepsilon}^{(p)}_{ij}, \tag{7.13}$$

and the equation of the yield surface is

$$\sigma'_{(2)} = A^2\{[B - (\sigma_{(1)}/\sqrt{3})]^2 - (B - C)^2\}. \tag{7.14}$$

In the space R' (7.14) represents one of the two shells (Fig. 8) of a hyperboloid of revolution with the asymptotic cone of Fig. 6. The yield stress in hydrostatic tension is $\sigma_{(1)} = C\sqrt{3}$; in simple shear it is $k = A(BC - C^2/2)^{1/2}$.

The cohesionless case is obtained by setting $C = 0$ or by restricting (7.11) to $\sigma_{(1)} \leqq 0$ and setting $A = 0$ for $\sigma_{(1)} = 0$.

Case 3: Another dissipation function is

$$\Phi = \pm 2[k - (\sigma_{(1)}/\sqrt{3})]\dot{\varepsilon}^{(p)}_1 \qquad (I, -I), \ldots, \tag{7.15}$$
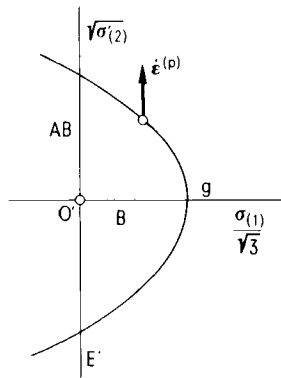

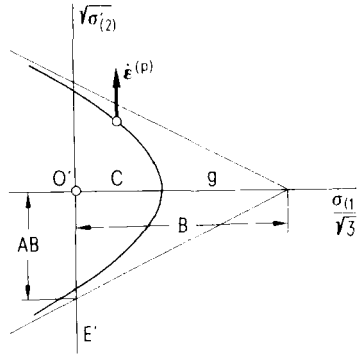FIG. 7. Longitudinal section of yield surface in case of VII,B.

FIG. 8.   Longitudinal section of yield surface in case of VII,B.

where

$$\sigma_{(1)} \leqq k\sqrt{3} \qquad (7.16)$$

and I, $-$I, $\ldots$, are the sectors obtained from Fig. 3 provided that the notations $d_1, \ldots$, are replaced by $\dot{\varepsilon}_1^{(p)}, \ldots$. The reasoning following (6.5) now yields

$$\sigma'_{I} = \tfrac{4}{3}[k - (\sigma_{(1)}/\sqrt{3})], \qquad \sigma'_{II} = \sigma'_{III} = -\tfrac{2}{3}[k - (\sigma_{(1)}/\sqrt{3})], \quad (7.17)$$

that is, (6.7) with k replaced by the expression between parentheses. For $\sigma_{(1)} = 0$, (7.17) reduces to (6.7); yield locus is therefore the Tresca hexagon of Fig. 2. For nonvanishing values of $\sigma_{(1)}$, (7.17) supplies yield hexagons whose linear dimensions are proportional to $k - \sigma_{(1)}/\sqrt{3}$; yield surface is thus the regular hexagonal pyramid considered by Drucker (1953). Its axis is g; its intersection with the deviatoric plane E' is the hexagon of Fig. 2; and its vertex is the point $\sigma_{(1)}/\sqrt{3} = k$ on g, that is, the point with coordinates $(k/\sqrt{3})(1, 1, 1)$.

Case 4: Let us finally consider the dissipation function

$$\Phi = A'\left(B - \frac{\sigma_{(1)}}{\sqrt{3}}\right)\dot{\varepsilon}_1^{(p)} \quad (1), \qquad \Phi = -A''\left(B - \frac{\sigma_{(1)}}{\sqrt{3}}\right)\dot{\varepsilon}_1^{(p)} \quad (-1), \ldots,$$
$$(7.18)$$

where A', A'', B are positive constants and where I, $-$I, $\ldots$, are sectors (Fig. 9) containing the projected positive and negative axes $\dot{\varepsilon}_1^{(p)}, \ldots$, respectively. Since Φ must be positive definite, we require

$$\sigma_{(1)} \leqq B\sqrt{3}, \qquad (7.19)$$

and we further assume that

$$A' < A'', \qquad (7.20)$$

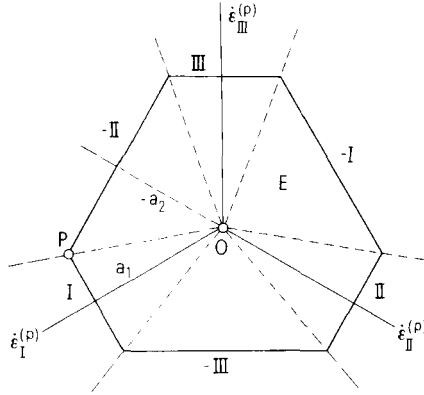a condition that will be motivated in connection with (7.30). On account

FIG. 9.   Sectors corresponding to the six definitions of Φ in (7.18).

of (A.15) Φ is constant on hexagons in the deviatoric plane E, having three axes of symmetry. Their convexity requires that $a_1/2 < -a_2$, that is,

$$(a_1/2) + a_2 < 0. \tag{7.21}$$

Since Φ must be continuous along the boundaries between the various sectors, for example, on OP, we have $A'\dot{\varepsilon}_I^{(p)} = -A''\dot{\varepsilon}_{II}^{(p)}$ along this boundary or, on account of (A.15),

$$A'a_1 + A''a_2 = 0. \tag{7.22}$$

It follows from (7.21) and (7.22) that

$$2A' > A''. \tag{7.23}$$

Applying the orthogonality condition

$$\sigma'_{ij} = (\partial/\partial\dot{\varepsilon}_{ij}^{(p)})(\Phi + \gamma'\dot{\varepsilon}_{(1)}^{(p)}) \tag{7.24}$$

to the open sector I and determining the multiplier γ' by the condition that σ'_{ij} is a deviator, we obtain

$$\sigma'_1 = \tfrac{2}{3}A'(B - \sigma_{(1)}/\sqrt{3}), \qquad \sigma'_{11} = \sigma'_{111} = -\tfrac{1}{3}A'(B - \sigma_{(1)}/\sqrt{3}). \tag{7.25}$$

In sector $-$I, A' must be replaced by $-A''$.

In the deviatoric stress plane (Fig. 10), (7.25) yields the single point I' as the image of sector I in Fig. 9. According to (A.15), its distance from O' and the distance of the image $-II'$ of sector $-$II from O' are

$$b_1 = \sqrt{\tfrac{2}{3}}\,A'B, \qquad -b_2 = \sqrt{\tfrac{2}{3}}\,A''B, \tag{7.26}$$

respectively. These points and the corresponding ones on the other projected axes define a hexagon with three axes of symmetry, convex since
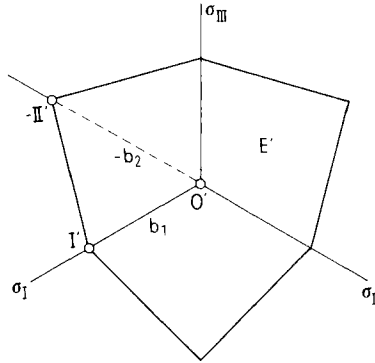
$$b_1 + (b_2/2) > 0 \tag{7.27}$$

FIG. 10.   Yield locus corresponding to Shield's pyramid for $\sigma_{(1)} = 0$.

on account of (7.26) and (7.23). In order to see that the sides of the hexagon in Fig. 10 are the images of the rays dividing the sectors in Fig. 9, we note that (7.22) may be interpreted as the equation of the ray OP. Using it as an additional side condition, we replace (7.24) by

$$\sigma'_{ij} = (\partial/\partial\dot{\varepsilon}^{(p)}_{ij})[\Phi + \gamma'\dot{\varepsilon}^{(p)}_{(1)} + \gamma''(A'\dot{\varepsilon}^{(p)}_{1} + A''\dot{\varepsilon}^{(p)}_{11})] \tag{7.28}$$

and obtain

$$\sigma'_{1} = \tfrac{2}{3}A'(B - \sigma_{(1)}/\sqrt{3}) + \tfrac{1}{3}\gamma''(2A' - A''),$$
$$\sigma'_{11} = -\tfrac{1}{3}A'(B - \sigma_{(1)}/\sqrt{3}) + \tfrac{1}{3}\gamma''(2A'' - A'), \tag{7.29}$$
$$\sigma'_{111} = -\tfrac{1}{3}A'(B - \sigma_{(1)}/\sqrt{3}) - \tfrac{1}{3}\gamma''(A' + A'')$$

in place of (7.25). With $\sigma_{(1)} = 0$, (7.29) is the parametric representation of a straight line in Fig. 10, and by means of (A.15) it is easy to see that it contains points I' for $\gamma'' = 0$ and $-$II' for $\gamma'' = -B$. Yield locus for $\sigma_{(1)} = 0$ is the hexagon of Fig. 10, having three axes of symmetry, and the yield surface for nonvanishing values of $\sigma_{(1)}$ is the pyramid intersecting E' in this hexagon and with vertex at $\sigma_{(1)} = B\sqrt{3}$ on the axis g.

Let the constants in (7.18) be determined by

$$A' = \frac{2\sqrt{3}\sin\varphi'}{3 + \sin\varphi'}, \qquad A'' = \frac{2\sqrt{3}\sin\varphi'}{3 - \sin\varphi'}, \qquad B = \sqrt{3}c\cot\varphi', \tag{7.30}$$

where c is a positive constant and φ' an acute angle. With (7.30), the inequalities (7.20) and (7.23) for A' and A'' are obviously satisfied, and the yield surface becomes the pyramid proposed by Shield (1955) as the only correct interpretation of Coulomb's law. In fact, it is easy to see that, with (7.30), the total stresses corresponding to (7.29) satisfy the condition

$$\sigma_{1} - \sigma_{11} + (\sigma_{1} + \sigma_{11})\sin\varphi' = 2c\cos\varphi', \tag{7.31}$$

which represent Coulomb's law

$$\tau = c - \sigma\tan\varphi' \tag{7.32}$$

on the straight line connecting the points I′ and −II′ in Fig. 10. For the other rays dividing sectors in Fig. 10 the proof is analogous.

C. Construction of the Dissipation Function

It has been noted that the materials considered in the two preceding sections do not satisfy the normality condition in stress space. However, they obey a restricted normality condition.

In the space R', the hydrostatic stress $\sigma_{(1)}$ is represented by a point P' on g. Once $\sigma_{(1)}$ is prescribed, the end point of the vector $\boldsymbol{\sigma}$ lies in a plane E'' passing through P' and parallel to E'. The deviator $\boldsymbol{\sigma}'$ is the projection of $\boldsymbol{\sigma}$ onto this plane. The yield surface intersects the plane E'' corresponding to $\sigma_{(1)}$ in a curve that may be considered the yield locus for the given value of $\sigma_{(1)}$. In cases 1 through 3 of Section VII,A and cases 1 and 2 of Section VII,B, the yield surface is a surface of revolution with axis g; the yield locus in any plane E'' is therefore a circle about P'. On account of Eqs. (7.4), (7.9), and (7.13), written in principal values, the vector $\boldsymbol{\varepsilon}^{(p)}$ has the direction of $\boldsymbol{\sigma}'$ in all these cases and hence obeys the normality condition with respect to the yield locus in the plane E''. Comparison of Figs. 2 and 3 and of Figs. 10 and 9 shows that the same is true in the remaining cases 3 and 4 of Section VII,B.

The models just mentioned are not the only ones proposed in literature. Other materials [see, e.g., Chen and Saleeb (1982)] pose the problem already discussed in Section VI,B: to find the dissipation function corresponding to a given yield condition. The solution is similar to the one given there. Let

$$F(\sigma'_{ij}, \sigma_{(1)}) = 0 \qquad [F(\sigma'_{ij} = 0) < 0] \qquad (7.33)$$

be the equation of the yield surface, with cross sections that are convex and star shaped with respect to their points P'. Postulating that the plastic strain rate is a deviator and that it obeys the restricted normality condition in any plane E'', we have

$$\dot{\varepsilon}^{(p)}_{ij} = \nu(\partial F/\partial \sigma'_{ij}) \qquad (\nu \geqq 0). \qquad (7.34)$$

From here on the reasoning following (6.10), with E replaced by E'' and $\mathbf{d}$ by $\dot{\boldsymbol{\varepsilon}}^{(p)}$, shows that and how a single-valued function $\Phi(\dot{\varepsilon}^{(p)}_{(2)}, \dot{\varepsilon}^{(p)}_{(3)}, \sigma_{(1)})$, homogeneous of the first degree in the plastic strain rates, can be constructed.

D. Coupled Elastic and Plastic Deformations

In all of the models discussed so far in Sections VI and VII, it is possible to decompose the total strain into plastic and elastic contributions, $\varepsilon^{(p)}_{ij} = \alpha_{ij}$

and $\varepsilon_{ij}^{(e)} = \varepsilon_{ij} - \alpha_{ij}$, respectively. In those cases where, as in (6.11), the free energy can be expressed in terms of the elastic strains and the dissipation function in terms of the plastic strain rates, it is obvious that the elastic and the plastic deformations are independent. In the case of the hardening material of Section VI,D the free energy (6.22) also contains the plastic strains. However, the connection between stress and elastic strain is still given by the generalized Hooke's law (6.13), and (6.25) connects the stress with the plastic strain and its time rate. In the soils treated so far, the dissipation function contains also the (elastic) dilatation. However, Eq. (6.13) is still valid and allows one to replace $\varepsilon_{(1)}$ by $\sigma_{(1)}$ in the dissipation function so that the orthogonality condition supplies relations like (7.4), (7.9), and (7.13) connecting the stress with the plastic strain rate. It follows that in all these examples elastic and plastic deformations can be obtained separately.

Houlsby (1979, 1981b) has pointed out that, in certain geological materials, the plastic deformation alters the elastic properties, so that the two types of deformations are coupled. As an example, he mentions the case where the shear modulus is a function of the elastic strain.

The simplest model of this type is defined by the modification

$$\Psi = (\lambda/2)\varepsilon_{(1)}^2 + (\mu + \nu\alpha_{(2)})(\varepsilon - \alpha)_{(2)} \tag{7.35}$$

of the free energy (7.1), where ν is a constant, and by the dissipation function (6.19). The external stress is quasi-conservative and given by

$$\sigma_{ij} = \lambda\varepsilon_{(1)}\delta_{ij} + 2(\mu + \nu\alpha_{(2)})(\varepsilon_{ij} - \alpha_{ij}) \tag{7.36}$$

in place of (6.12). The generalization of Hooke's law now reads

$$\sigma_{ij} = \lambda\varepsilon_{(1)}^{(e)}\delta_{ij} + 2(\mu + \nu\varepsilon_{(2)}^{(p)})\varepsilon_{ij}^{(e)}; \tag{7.37}$$

besides, (7.36) yields

$$\sigma'_{ij} = 2(\mu + \nu\alpha_{(2)})(\varepsilon'_{ij} - \alpha_{ij}). \tag{7.38}$$

The quasi-conservative part of the internal stress is

$$\beta_{ij}^{(q)} = -2(\mu + \nu\alpha_{(2)})(\varepsilon_{ij} - \alpha_{ij}) + 2\nu(\varepsilon - \alpha)_{(2)}\alpha_{ij} + \gamma'\delta_{ij} \tag{7.39}$$

in place of (6.15). From (7.39) and (7.38) it follows that

$$\beta_{ij}^{(q)\prime} = -\sigma'_{ij} + 2\nu\varepsilon_{(2)}^{(e)}\varepsilon_{ij}^{(p)}, \tag{7.40}$$

and from (6.19) we obtain

$$\beta_{ij}^{(d)\prime} = k(\tfrac{1}{2}\dot{\varepsilon}_{(2)}^{(p)})^{-1/2}\dot{\varepsilon}_{ij}^{(p)}. \tag{7.41}$$

Equation (2.35) now yields the relation

$$\sigma'_{ij} = 2\nu\varepsilon_{(2)}^{(e)}\varepsilon_{ij}^{(p)} + k(\tfrac{1}{2}\dot{\varepsilon}_{(2)}^{(p)})^{-1/2}\dot{\varepsilon}_{ij}^{(p)} \tag{7.42}$$

and finally the yield locus

$$(\sigma' - 2\nu\varepsilon_{(2)}^{(e)}\varepsilon^{(p)})_{(2)} = 2k^2. \tag{7.43}$$

The two constitutive equations (7.37) and (7.42) may be considered as generalizations of (6.13) and (6.21) or (6.27), respectively. Each of them contains a term that establishes coupling between the elastic and plastic deformations.

Expression (7.35) seems to be equivalent to the free energy of Houlsby (1981b). In a former paper (1979) he used a slightly different expression

$$\Psi = (\lambda/2 + \mu/3)\varepsilon_{(1)}^2 + (\mu + \nu\alpha_{(2)})(\varepsilon - \alpha)_{(2)}. \tag{7.44}$$

In another publication Houlsby (1980) also derived the so-called "modified Cam–Clay" model of Schofield and Wroth (1968) from appropriate functions Ψ and Φ.

VIII. Viscoplasticity

A. CREEP OF METALS

Let us return to the material defined by the governing functions (6.11), dropping, however, the condition that Φ be homogeneous of the first degree. Here, the internal strain tensor α_{ij} becomes what is usually called the viscous strain $\varepsilon_{ij}^{(v)}$, and the difference $\varepsilon_{ij}^{(e)} = \varepsilon_{ij} - \varepsilon_{ij}^{(v)}$ is again the elastic strain.

The externl stress is quasi-conservative and given by (6.12). Stress and elastic strain are connected by (6.13) or (6.14), and the quasi-conservative part of the internal stress is supplied by (6.15). Equation (6.16) is still valid; combined with (2.35) it yields $\sigma_{ij}' = \beta_{ij}^{(d)'}$. Since the dissipative internal stress obeys relations that are analogous to (5.36), (5.37), we have

$$\sigma_{ij}' = \nu[2(\partial\Phi/\partial\dot\alpha_{(2)})\dot\alpha_{ij} + 3(\partial\Phi/\partial\dot\alpha_{(3)})(\dot\alpha_{ik}\dot\alpha_{kj} - \tfrac{1}{3}\dot\alpha_{(2)}\delta_{ij})], \tag{8.1}$$

where

$$\nu = \Phi(2(\partial\Phi/\partial\dot\alpha_{(2)})\dot\alpha_{(2)} + 3(\partial\Phi/\partial\dot\alpha_{(3)})\dot\alpha_{(3)})^{-1}. \tag{8.2}$$

The simplest special case is the one where the dissipation function depends on $\alpha_{(2)} = \dot\varepsilon_{(2)}^{(v)}$ alone. In the deviatoric plane E of the system $\dot\varepsilon_1^{(v)}, \ldots$, the function Φ is then constant on circles about the origin. Equations (8.1) and (8.2) reduce to

$$\sigma_{ij}' = 2\nu(\partial\Phi/\partial\dot\varepsilon_{(2)}^{(v)})\dot\varepsilon_{ij}^{(v)}, \qquad \nu = \Phi[2(\partial\Phi/\partial\dot\varepsilon_{(2)}^{(v)})\dot\varepsilon_{(2)}^{(v)}]^{-1} \tag{8.3}$$

or, equivalently, to the single relation (5.48),

$$\sigma_{ij}' = (\Phi/\dot\varepsilon_{(2)}^{(v)})\dot\varepsilon_{ij}^{(v)}. \tag{8.4}$$

Let us specialize further, assuming that

$$\Phi = 2\mu'(\dot{\varepsilon}_{(2)}^{(v)})^n, \tag{8.5}$$

where μ' is a coefficient (possibly dependent on ϑ). This dissipation function is homogeneous of degree $2n$. With $n = 1$ it is analogous to the dissipation function (5.46) of the incompressible Newtonian liquid; with $n = \frac{1}{2}$, a case we will exclude in what follows, it reduces to (6.19), that is, to the dissipation function of a plastic material of the v. Mises type. With (8.5) the deviatoric stress (8.4) becomes

$$\sigma'_{ij} = 2\mu'(\dot{\varepsilon}_{(2)}^{(v)})^{n-1}\dot{\varepsilon}_{ij}^{(v)}, \tag{8.6}$$

and by comparison with (5.45) we see that the viscous response is the one of a quasi-linear liquid. From (8.6) we obtain

$$\sigma'_{(2)} = 4\mu'^{2}(\dot{\varepsilon}_{(2)}^{(v)})^{2n-1} \tag{8.7}$$

and hence the inversion of (8.6),

$$\dot{\varepsilon}_{ij}^{(v)} = (\sigma'_{(2)}/4\mu'^{2})^{1-n/2n-1}(\sigma'_{ij}/2\mu'). \tag{8.8}$$

On account of (8.5) and (8.7), Φ' in the space R' is constant on circular cylinders with axis g, and (8.8), written in principal values, establishes orthogonality in R'. With

$$(1 - n)/(2n - 1) =: m - 1 \quad \text{and} \quad (2\mu')^{1/1-2n} =: k \tag{8.9}$$

(8.8) assumes the form

$$\dot{\varepsilon}_{ij}^{(v)} = k(\sigma'_{(2)})^{m-1}\sigma'_{ij}. \tag{8.10}$$

This is Odqvist's equation (1934; see also 1966) for secondary creep of incompressible materials, a generalization of the so-called Norton's law (1929).

Another special case is obtained if one replaces (8.5) by

$$\Phi = A(\dot{\varepsilon}_1^{(v)2})^n \quad (\mathrm{I}, -\mathrm{I}), \dots, \tag{8.11}$$

where A is a positive scalar (possibly dependent on ϑ) and where the roman numerals $\mathrm{I}, -\mathrm{I}, \dots$, refer to the six sectors of Fig. 3, provided that $d_\mathrm{I}, \dots$, are replaced by $\dot{\varepsilon}_1^{(v)}, \dots$. The function Φ is constant on the regular hexagons mentioned in connection with (6.5). In the open sectors I and $-\mathrm{I}$ the deviatoric stress becomes

$$\sigma'_\mathrm{I} = \tfrac{2}{3}A(\dot{\varepsilon}_1^{(v)2})^{n-1}\dot{\varepsilon}_1^{(v)}, \qquad \sigma'_\mathrm{II} = \sigma'_\mathrm{III} = -\tfrac{1}{3}A(\dot{\varepsilon}_1^{(v)2})^{n-1}\dot{\varepsilon}_1^{(v)}. \tag{8.12}$$

The images of I and $-\mathrm{I}$ are thus the projected axes $\sigma_\mathrm{I}, -\sigma_\mathrm{I}$, respectively, in Fig. 2, and similar statements hold for the other sectors. The line dividing the sectors I and $-\mathrm{II}$ in Fig. 3 is mapped onto the open sector between the projected axes σ_I and $-\sigma_\mathrm{II}$ in Fig. 2, and in this sector, as well as the one

between the projected axes $-\sigma_{\text{I}}$ and σ_{II}, (8.12) has the inversions

$$\dot{\varepsilon}_{\text{I}}^{(v)} = -\dot{\varepsilon}_{\text{II}}^{(v)} = (3/2A)^{1/2n-1}(\sigma_{\text{I}}'^2)^{1-n/2n-1}\sigma_{\text{I}}', \qquad \dot{\varepsilon}_{\text{III}}^{(v)} = 0. \qquad (8.13)$$

In the remaining sectors in Figs. 3 and 2 similar expressions are valid.

B. Elastic, Viscoplastic Materials

The free energy and the dissipation function given by (6.11) may also be used to define elastic, viscoplastic materials. As shown in the preceding section, stress and elastic strain are connected by (6.13) or (6.14) and the stress deviator follows from (8.1) and (8.2).

Let us first use the special form (8.5) of the dissipation function (Z.17.1). It has been noted that it corresponds for $n = 1$ to a modified Newtonian liquid and for $n = \frac{1}{2}$ to an elastic, perfectly plastic material of the v. Mises type. Elastic, viscoplastic materials may be characterized by an exponent n that is slightly larger than $\frac{1}{2}$. To show this, let us note that, in principal axes, (8.6) assumes the form

$$\sigma_{\text{I}}' = 2\mu'(\dot{\varepsilon}_{(2)}^{(v)})^{n-1}\dot{\varepsilon}_{\text{I}}^{(v)}, \ldots \qquad (8.14)$$

In the case of uniaxial stress σ_{I} we have $\sigma_{\text{I}}' = (2/3)\sigma_{\text{I}}$ and $\dot{\varepsilon}_{(2)}^{(v)} = (3/2)\dot{\varepsilon}_{\text{I}}^{(v)2}$ in analogy to (6.33) and (6.34). Thus, (8.14) yields

$$\sigma_{\text{I}} = 2\mu'(\tfrac{3}{2})^n(\dot{\varepsilon}_{\text{I}}^{(v)2})^{n-1}\dot{\varepsilon}_{\text{I}}^{(v)}. \qquad (8.15)$$

Figure 11, which, for negative values of $\dot{\varepsilon}_{\text{I}}^{(v)}$, is to be reflected at the origin,
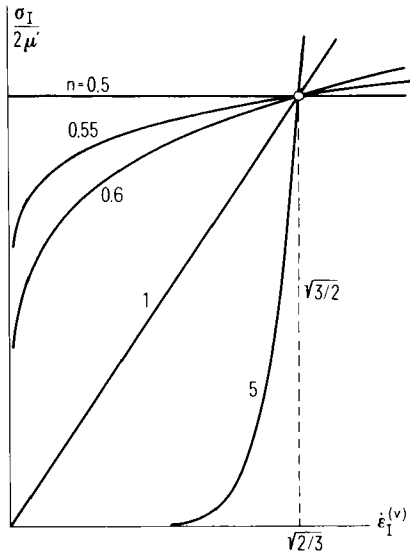

FIG. 11. Response of viscoplastic materials in simple tension.

shows $\sigma_1/2\mu'$ as a function of $\dot{\varepsilon}_1^{(v)}$ for a few values of n, including the Newtonian and the perfectly plastic cases. For $\frac{1}{2} < n < 1$ the curves leave the origin with a vertical tangent. For $n \to \frac{1}{2}$ they approach the vertical axis and the horizontal $\sigma_1/2\mu' = \sqrt{3/2}$, that is, the perfectly plastic response.

Another choice of the dissipation function is

$$\Phi = 2a\dot{\varepsilon}_{(2)}^{(v)} + k(2\dot{\varepsilon}_{(2)}^{(v)})^{1/2}, \tag{8.16}$$

where a and k are scalars. The corresponding stress deviator (8.4) is

$$\sigma'_{ij} = [2a + k(\tfrac{1}{2}\dot{\varepsilon}_{(2)}^{(v)})^{-1/2}]\dot{\varepsilon}_{ij}^{(v)}. \tag{8.17}$$

This is the constitutive equation proposed by Hohenemser and Prager (1932) for viscoplastic materials of the Bingham type (1922).

Let us finally note that the dissipation function (6.30) exhibits a certain similarity to (8.5). This similarity appears also in Figs. 5 and 11. The principal difference is that the curves of Fig. 5 start from the point $\sigma_1/\sigma_0 = 1$, whereas those of Fig. 11 start at the origin. Materials with rate-dependent yield thus have a definite yield stress for vanishing strain rate, whereas the stress of the material considered in this section tends to zero with $\dot{\varepsilon}_1^{(v)} \to 0$. It is questionable whether this difference is practically observable.

Germain *et al.* (1983) mention a few more general viscoplastic materials and an application to damage of ductile materials.

Returning to Fig. 11 we observe that, for $n > 1$, the curves (as the one displayed for $n = 5$) leave the origin with a horizontal tangent. With increasing n they approach the horizontal axis and the vertical $\dot{\varepsilon}_1^{(v)} = \sqrt{2/3}$. Thus, the dissipation function (8.5) may also be used for materials that tend to the *locking material* described by Prager (1957).

IX. Viscoelasticity

A. Linear Viscoelasticity

The response in pure tension of the materials treated in Sections IV through VIII can be modeled by simple combinations of springs, dashpots, and other simple elements as described, for example, by Lee (1962). On the other hand, such models, suggested by the results of tension tests, may be used to establish the general response of more complicated materials.

The description of viscoelastic materials can be based on the Maxwell grid (Fig. 12), where certain elements might be dropped but none of the Maxwell elements is to be replaced by a single dashpot if impact response is to be ensured (Z.18.1). (The optional single spring allows modeling of solids as well as fluids.) To simulate the actual response in simple tension, it is usually necessary to introduce quite a number of internal parameters
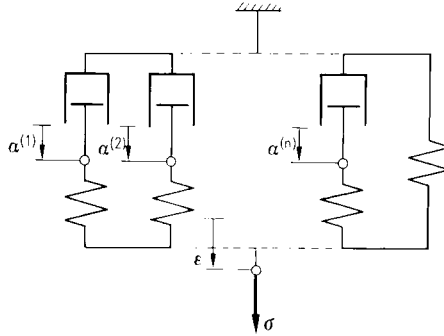
FIG. 12. The Maxwell grid.

$\alpha^{(1)}$, $\alpha^{(2)}$, ..., $\alpha^{(n)}$ besides the external extension ε. The generalization for arbitrary deformations requires a set of internal strain tensors $\alpha_{ij}^{(r)}$ ($r = 1, 2, ..., n$) beside the external strain ε_{ij}.

The linear case corresponds to quadratic functions Ψ and Φ. If we confine ourselves to the isotropic case, readmitting thermal processes, the governing functions (Z.18.2) are generalizations of (4.17) and (5.27). The free energy may be defined by

$$\Psi = \rho\psi = \Psi_0 - S_0(\vartheta - \vartheta_0) + \sum_{r=1}^{n+1} \left[\frac{\lambda^{(r)}}{2}(\varepsilon - \alpha^{(r)})_{(1)}^2 + \mu^{(r)}(\varepsilon - \alpha^{(r)})_{(2)} \right]$$

$$- (\vartheta - \vartheta_0) \sum_{r=1}^{n+1} (3\lambda^{(r)} + 2\mu^{(r)})\kappa^{(r)}(\varepsilon - \alpha^{(r)})_{(1)} - \frac{\rho c}{2\vartheta_0}(\vartheta - \vartheta_0)^2,$$

$$(9.1)$$

where ρ is constant and $\alpha_{ij}^{(n+1)} = 0$, and the dissipation function by

$$\Phi = \rho\varphi = \sum_{r=1}^{n} (\lambda^{(r)\prime}\dot{\alpha}_{(1)}^{(r)2} + 2\mu^{(r)\prime}\dot{\alpha}_{(2)}^{(r)}) + \frac{\gamma}{\vartheta}q_{(1)}. \tag{9.2}$$

The entropy might be obtained from (9.1) by means of (2.13) and the internal energy subsequently from Eq. (2.3), modified by addition of the arguments $\alpha_{ij}^{(r)}$ besides ε_{ij} and ϑ. Since (9.2) does not contain $\dot{\varepsilon}_{ij}$, the external stress is quasi-conservative and given by

$$\sigma_{ij} = \frac{\partial\Psi}{\partial\varepsilon_{ij}} = \sum_{r=1}^{n+1} [\lambda^{(r)}(\varepsilon - \alpha^{(r)})_{(1)}\delta_{ij} + 2\mu^{(r)}(\varepsilon_{ij} - \alpha_{ij}^{(r)})]$$

$$- (\vartheta - \vartheta_0)\delta_{ij} \sum_{r=1}^{n+1} (3\lambda^{(r)} + 2\mu^{(r)})\kappa^{(r)}. \tag{9.3}$$

The internal stresses are

$$\beta_{ij}^{(qr)} = \frac{\partial\Psi}{\partial\alpha_{ij}^{(r)}} = -\lambda^{(r)}(\varepsilon - \alpha^{(r)})_{(1)}\delta_{ij} - 2\mu^{(r)}(\varepsilon_{ij} - \alpha_{ij}^{(r)})$$

$$+ (3\lambda^{(r)} + 2\mu^{(r)})\kappa^{(r)}(\vartheta - \vartheta_0)\delta_{ij}, \tag{9.4}$$

$$\beta_{ij}^{(dr)} = \frac{1}{2} \frac{\partial \Phi}{\partial \dot{\alpha}_{ij}^{(r)}} = \lambda^{(r)\prime} \dot{\alpha}_{(1)}^{(r)} \delta_{ij} + 2\mu^{(r)\prime} \dot{\alpha}_{ij}^{(r)}, \tag{9.5}$$

where the fact has been used that $\nu = \frac{1}{2}$ since Φ is quadratic in $\dot{\alpha}_{ij}^{(r)}$. Application of the orthogonality condition to q_i yields

$$- \vartheta_{,i}/\vartheta = \tfrac{1}{2}(\partial\Phi/\partial q_i) = (\gamma/\vartheta)q, \tag{9.6}$$

and hence Fourier's law (3.11). Equation (2.17), applied to the various internal stresses (9.4) and (9.5), supplies the differential equations

$$\lambda^{(r)\prime} \dot{\alpha}_{(1)}^{(r)} \delta_{ij} + 2\mu^{(r)\prime} \dot{\alpha}_{ij}^{(r)} = [\lambda^{(r)}(\varepsilon - \alpha^{(r)})_{(1)} - (3\lambda^{(r)} + 2\mu^{(r)})$$
$$\times \kappa^{(r)}(\vartheta - \vartheta_0)]\delta_{ij} + 2\mu^{(r)}(\varepsilon_{ij} - \alpha_{ij}^{(r)}). \tag{9.7}$$

The problem that remains is the elimination of the internal parameters from (9.3) and the n equations (9.7). To solve it, it is convenient to decompose these relations into their deviatoric and isotropic parts, obtaining

$$\sigma_{ij}' = 2 \sum_{r=1}^{n+1} \mu^{(r)}(\varepsilon_{ij}' - \alpha_{ij}^{(r)\prime}),$$

$$\sigma_{(1)} = 3 \sum_{r=1}^{n+1} K^{(r)}[(\varepsilon - \alpha^{(r)})_{(1)} - 3\kappa^{(r)}(\vartheta - \vartheta_0)] \qquad (\alpha_{ij}^{(n+1)} = 0) \tag{9.8}$$

and

$$\mu^{(r)\prime} \dot{\alpha}_{ij}^{(r)\prime} = \mu^{(r)}(\varepsilon_{ij}' - \alpha_{ij}^{(r)\prime}),$$
$$K^{(r)\prime} \dot{\alpha}_{(1)}^{(r)} = K^{(r)}[(\varepsilon - \alpha^{(r)})_{(1)} - 3\kappa^{(r)}(\vartheta - \vartheta_0)], \qquad (r = 1, 2, \ldots, n) \tag{9.9}$$

where

$$K^{(r)} = \lambda^{(r)} + \tfrac{2}{3}\mu^{(r)} \qquad \text{and} \qquad K^{(r)} = \lambda^{(r)\prime} + \tfrac{2}{3}\mu^{(r)\prime} \tag{9.10}$$

are bulk moduli. If we differentiate the first equation (9.8) n times, expressing $\dot{\alpha}_{ij}^{(r)\prime}$ after each step in terms of $\varepsilon_{ij}' - \alpha_{ij}^{(r)\prime}$ by means of the first equation (9.9), we obtain the time derivatives $\dot{\sigma}_{ij}', \ddot{\sigma}_{ij}', \ldots, \sigma_{ij}^{(n)\prime}$ as linear functions of the $\dot{\varepsilon}_{ij}', \ddot{\varepsilon}_{ij}', \ldots, \varepsilon_{ij}^{(n)\prime}$ and the various $\varepsilon_{ij}' - \alpha_{ij}^{(r)\prime}$. Eliminating the n terms $\varepsilon_{ij}' - \alpha_{ij}^{(r)\prime}$ from the resulting $n + 1$ equations [the first equation (9.8) and its n derivatives], we are left with a single tensorial differential equation of the type

$$\sigma_{ij}' + p^{(1)\prime} \dot{\sigma}_{ij}' + \cdots + p^{(n)\prime} \sigma_{ij}^{(n)\prime} = q^{(0)\prime} \varepsilon_{ij}' + q^{(1)\prime} \dot{\varepsilon}_{ij}' + \cdots + q^{(n)\prime} \varepsilon_{ij}^{(n)\prime} \tag{9.11}$$

between the strain and stress deviators. If we further differentiate the second equation (9.8) n times, expressing $\dot{\alpha}_{(1)}^{(r)}$ after each step in terms of $(\varepsilon - \alpha^{(r)})_{(1)} - 3\kappa^{(r)}(\vartheta - \vartheta_0)$ by means of the second equation (9.9), we obtain the $\dot{\sigma}_{(1)}, \ddot{\sigma}_{(1)}, \ldots, \sigma_{(1)}^{(n)}$ as linear functions of $\dot{\varepsilon}_{(1)} - 3\kappa^{(r)}\dot{\vartheta}, \ddot{\varepsilon}_{(1)} - 3\kappa^{(r)}\dot{\vartheta}, \ldots,$ $\varepsilon_{(1)}^{(n)} - 3\kappa^{(r)}\vartheta^{(n)}$ $(r = 1, 2, \ldots, n + 1)$ and of $(\varepsilon - \alpha^{(r)})_{(1)} - 3\kappa^{(r)}(\vartheta - \vartheta_0)$ $(r = 1, 2, \ldots, n)$. Elimination of the last n expressions from the $n + 1$

equations we dispose of leaves a single scalar differential equation of the type

$$\sigma_{(1)} + p^{(1)}\dot{\sigma}_{(1)} + \cdots + p^{(n)}\sigma_{(1)}^{(n)} = q^{(0)}\varepsilon_{(1)} + q^{(1)}\dot{\varepsilon}_{(1)} + \cdots + q^{(n)}\varepsilon_{(1)}^{(n)}$$
$$+ r^{(0)}(\vartheta - \vartheta_0) + r^{(1)}\dot{\vartheta} + \cdots + r^{(n)}\vartheta^{(n)}.$$
(9.12)

Except for the thermal terms, (9.11) and (9.12) are the differential equations commonly used in texts on viscoelasticity [e.g., Flügge (1975)]. It is clear that the coefficients are not free but determined by the functions Ψ and Φ, that is, by $\lambda^{(r)}$, $\mu^{(r)}$, $\lambda^{(r)\prime}$, $\mu^{(r)\prime}$, and $\kappa^{(r)}$. A simple example and a generalization for nonlinear response are given elsewhere (Z.18.2).

The case where the material is *free of bulk viscosity* is obtained by introducing the internal strains $\alpha_{ij}^{(r)}$ as deviators. The second equation (9.9) must then be dropped since the left-hand side becomes indeterminate. The second equation (9.8) reduces to

$$\sigma_{(1)} = 3 \sum_{r=1}^{n+1} K^{(r)}[\varepsilon_{(1)} - 3\kappa^{(r)}(\vartheta - \vartheta_0)], \qquad (9.13)$$

a degenerate form of (9.12), connecting the isotropic parts of ε_{ij} and σ_{ij} with the temperature increase. The remaining equations (9.8) and (9.9) become

$$\sigma_{ij}' = 2 \sum_{r=1}^{n+1} \mu^{(r)}(\varepsilon_{ij}' - \alpha_{ij}^{(r)}), \qquad \mu^{(r)\prime}\dot{\alpha}_{ij}^{(r)} = \mu^{(r)}(\varepsilon_{ij}' - \alpha_{ij}^{(r)}), \qquad (9.14)$$

and the process expounded following (9.10) supplies a differential equation of the type (9.11) for the deviatoric parts of ε_{ij} and σ_{ij}.

If *thermal effects* can be *neglected*, the terms with $\vartheta - \vartheta_0$ in (9.1) and with $q_{(1)}$ in (9.2) are to be dropped. Thus, the terms containing $\vartheta - \vartheta_0$ disappear from (9.8) and (9.9) as well as (9.13), and the coefficients $r^{(0)}$, $r^{(1)}, \ldots, r^{(n)}$ in (9.12) become zero.

In an *incompressible* material $\varepsilon_{(1)}$ and the $\kappa^{(r)\prime}$ are zero, whereas the $K^{(r)}$ become infinite. Thus, (9.13) reduces to

$$\sigma_{(1)} = -3p, \qquad (9.15)$$

where p is an indeterminate hydrostatic pressure, and Eqs. (9.14) can be written

$$\sigma_{ij} = -p\delta_{ij} + 2 \sum_{r=1}^{n+1} \mu^{(r)}(\varepsilon_{ij} - \alpha_{ij}^{(r)}), \qquad \mu^{(r)\prime}\dot{\alpha}_{ij}^{(r)} = \mu^{(r)}(\varepsilon_{ij} - \alpha_{ij}^{(r)}). \qquad (9.16)$$

Elimination of the internal parameters now yields the differential equation (9.11), where the primes after ε_{ij} and its derivatives may be dropped.

B. Rivlin-Ericksen Liquids

A class of isotropic materials proposed and discussed by Rivlin and Ericksen (1955) is defined by the condition that the stress depends alone on the gradients of the displacement, velocity, acceleration, and higher accelerations up to a certain order, so that thermal effects, in particular, are absent. In the special case where the material is an incompressible liquid, Truesdell and Noll (1965) present the constitutive equations in the form

$$\sigma_{ij} = -p\delta_{ij} + F_{ij}(A_{pq}^{(1)}, A_{pq}^{(2)}, \ldots, A_{pq}^{(n)}), \qquad (9.17)$$

where $A_{ij}^{(r)}$ ($r = 1, 2, \ldots, n$) are the Rivlin–Ericksen tensors

$$A_{ij}^{(1)} = 2d_{ij}, \qquad A_{ij}^{(r+1)} = \dot{A}_{ij}^{(r)} + A_{ik}^{(r)} v_{k,j} + A_{kj}^{(r)} v_{k,i}, \qquad (9.18)$$

where $v_{j,i}$ is the velocity gradient and F_{ij} a deviatoric function.

If we confine ourselves to small displacements and displacement gradients, the tensors $A_{ij}^{(r)}$ reduce to $2\varepsilon_{ij}^{(r)}$ ($r = 1, 2, \ldots, n$), and the constitutive equation (9.17) reduces to (9.15) and

$$\sigma_{ij}' = F_{ij}(\dot{\varepsilon}_{ij}, \ddot{\varepsilon}_{ij}, \ldots, \varepsilon_{ij}^{(n)}). \qquad (9.19)$$

In view of Section IX,A it appears remarkable that the time derivatives of the strains are present in (9.19) up to the order n, whereas the stress derivatives are absent. In any event the question arises whether the Rivlin–Ericksen liquid can be obtained by the approach that has been quite successful in the preceding sections.

Let us note that, in the approximation leading to (9.19), the density is to be treated (Z.5.3) as constant. If we neglect thermal effects but retain the notion of internal parameters, we have to start from a free energy of the form

$$\Psi(\varepsilon_{ij}, \alpha_{ij}^{(r)}) \qquad (r = 1, 2, \ldots, n). \qquad (9.20)$$

On account of the orthogonality condition and of (2.17), the dissipation function depends on the time derivatives of exactly those internal parameters that appear as arguments of Ψ. Besides, it may depend on ε_{ij}, $\alpha_{ij}^{(r)}$ and possibly even on $\dot{\varepsilon}_{ij}$, so that

$$\Phi(\dot{\varepsilon}_{ij}, \dot{\alpha}_{ij}^{(r)}, \varepsilon_{ij}, \alpha_{ij}^{(r)}) \qquad (r = 1, 2, \ldots, n). \qquad (9.21)$$

The quasi-conservative stresses obtained from (9.20) by means of (2.33) and (2.34) are functions of the arguments of Ψ. The dissipative stresses depend on the arguments of Φ, no matter whether we calculate them by means of (2.27), (2.28) or (2.30), (2.31). According to (2.32) the external stress deviator assumes the form

$$\sigma_{ij}' = F_{ij}(\dot{\varepsilon}_{pq}, \dot{\alpha}_{pq}^{(r)}, \varepsilon_{pq}, \alpha_{pq}^{(r)}), \qquad (9.22)$$

and (2.35) yields n differential equations of the type

$$g_{ij}^{(s)}(\dot{\varepsilon}_{pq}, \dot{\alpha}_{pq}^{(r)}, \varepsilon_{pq}, \alpha_{pq}^{(r)}) = h_{ij}^{(s)}(\varepsilon_{pq}, \alpha_{pq}^{(r)}) \qquad (s = 1, 2, \ldots, n). \quad (9.23)$$

In order to arrive at (9.19) from (9.22) and (9.23) it must be possible to eliminate the internal parameters and their time derivatives. To do this, let us differentiate (9.22) and (9.23) n times with respect to time, obtaining altogether $(n + 1)^2$ equations for $\varepsilon_{pq}, \dot{\varepsilon}_{pq}, \ldots, \varepsilon_{pq}^{(n+1)}, \alpha_{pq}^{(r)}, \dot{\alpha}_{pq}^{(r)}, \ldots, \alpha_{pq}^{(n+1)(r)}$ and $\sigma_{ij}', \dot{\sigma}_{ij}', \ldots, \sigma_{ij}^{(n)\prime}$, that is, for $(n + 1)(n + 3)$ unknowns. Eliminating the $n(n + 2)$ internal parameters including their derivatives, we are left with a single equation containing $2n + 3$ unknowns, namely, ε_{pq} and σ_{ij}' with $n + 1$ and n derivatives, respectively. In other words, we obtain a differential equation of the type

$$G_{ij}(\sigma_{pq}', \dot{\sigma}_{pq}', \ldots, \sigma_{pq}^{(n)\prime}, \varepsilon_{pq}, \dot{\varepsilon}_{pq}, \ldots, \varepsilon_{pq}^{(n+1)}) = 0. \quad (9.24)$$

That the order of the operator G_{ij} is one higher in ε_{pq} than in σ_{pq} is due to the fact that, in the interest of generality and in contrast to Section IX,A, $\dot{\varepsilon}_{ij}$ has been included as an argument of the dissipation function (9.21).

The result (9.24) can be obtained without use of the orthogonality condition. In fact, (9.22) and (9.23) and hence (9.24) follow from the mere assumption that the dissipative stresses depend on the (internal and possibly external) strain rates and strains. In the exotic case of dissipative stresses containing higher-order time derivatives of $\alpha_{ij}^{(r)}$ and ε_{ij} an increased number of differentiations would lead to an equation of the type (9.24) with an operator of higher order.

On the other hand, it is in general impossible to reduce (9.24) to (9.19). Even in the linear version (9.11) of (9.24) this is not possible. Already in the case $n = 1$, treated in (Z.18.2), the coefficient $p^{(1)\prime}$ of $\dot{\sigma}_{ij}'$ does not vanish. It follows that the approach based on internal parameters and the constitutive equations proposed by Rivlin and Ericksen are incompatible whether the orthogonality condition be used or not.

X. Conclusion

The materials treated starting with Section III confirm what was stated in the introduction (Section I): an amazing number of constitutive relations of practical significance can be deduced from appropriately chosen expressions for the free energy and the dissipation function.

Of particular interest is the fact that orthogonality in velocity space, which is essentially responsible for the results, does not necessarily imply orthogonality in force space since there is in general, as noted in Section I, no duality between the two spaces. As a consequence, orthogonality in velocity space is apt to explain the actual behavior of soils (Section VII), where the

theory of the plastic potential and its justification by Drucker (1951) break down.

On the other hand, we have encountered a few cases where the orthogonality condition restricts the form of the constitutive equations generally used. They are (Section V,C) the Reiner–Rivlin liquid, where the scalar functions multiplying d_{ij} and $d_{ik}d_{kj} - (d_{(2)}/3)$ in (5.44) are not free but determined by the dissipation function, and the quasilinear liquid, where the viscosity function μ' in (5.45) must be independent of $d_{(3)}$.

Another exception is the Rivlin–Ericksen liquid (Section IX,B). Here, the approach based on internal parameters supplies the constitutive equation (9.24) instead of (9.19) no matter whether maximal rate of entropy production is used or not.

Appendix

1. Let v_i be a vector and t_{ij} be a symmetric tensor of order 2 in an orthogonal Cartesian coordinate system. The only basic invariant of v_i may be defined by

$$v_{(1)} = v_i v_i, \tag{A.1}$$

the basic invariants of t_{ij} by

$$t_{(1)} = t_{ii}, \qquad t_{(2)} = t_{ij}t_{ji}, \qquad t_{(3)} = t_{ij}t_{jk}t_{ki}, \tag{A.2}$$

and the mixed invariants by

$$m_{(1)} = v_i t_{ij} v_j, \qquad m_{(2)} = v_i t_{ij} t_{jk} v_k. \tag{A.3}$$

The invariants of the deviator

$$t'_{ij} = t_{ij} - \tfrac{1}{3} t_{(1)} \delta_{ij} \tag{A.4}$$

are

$$t'_{(1)} = 0, \qquad t'_{(2)} = t_{(2)} - \tfrac{1}{3} t_{(1)}^2, \qquad t'_{(3)} = t_{(3)} - t_{(2)} t_{(1)} + (2/g) t_{(1)}^3. \tag{A.5}$$

Differentiating the invariants, one obtains

$$
\begin{aligned}
&\partial v_{(1)}/\partial v_i = 2v_i, \\
&\partial t_{(1)}/\partial t_{ij} = \delta_{ij}, \qquad \partial t_{(2)}/\partial t_{ij} = 2t_{ij}, \qquad \partial t_{(3)}/\partial t_{ij} = 3t_{ik}t_{kj}, \\
&\partial m_{(1)}/\partial v_i = 2t_{ij}v_j, \qquad \partial m_{(2)}/\partial v_i = 2t_{ij}t_{jk}v_k, \\
&\partial m_{(1)}/\partial t_{ij} = v_i v_j, \qquad \partial m_{(2)}/\partial t_{ij} = (v_i t_{jk} + v_j t_{ik})v_k.
\end{aligned}
\tag{A.6}
$$

2. In an isotropic material a function Φ of q_i and d_{jk} depends only on the invariants of q_i and d_{jk} (and possibly on certain independent state

variables). Thus, Φ has the form

$$\Phi(q_{(1)}, d_{(1)}, d_{(2)}, d_{(3)}, m_{(1)}, m_{(2)}). \tag{A.7}$$

On account of (A.6) we then have

$$\frac{\partial \Phi}{\partial q_i} = 2\left(\frac{\partial \Phi}{\partial q_{(1)}} q_i + \frac{\partial \Phi}{\partial m_{(1)}} d_{ij} q_j + \frac{\partial \phi}{\partial m_{(2)}} d_{ij} d_{jk} q_k\right),$$

$$\frac{\partial \Phi}{\partial d_{ij}} = \frac{\partial \Phi}{\partial d_{(1)}} \delta_{ij} + 2\frac{\partial \Phi}{\partial d_{(2)}} d_{ij} + 3\frac{\partial \Phi}{\partial d_{(3)}} d_{ik} d_{kj} \tag{A.8}$$

$$+ \frac{\partial \Phi}{\partial m_{(1)}} q_i q_j + \frac{\partial \Phi}{\partial m_{(2)}}(q_i d_{jk} + q_j d_{ik}) q_k,$$

and hence

$$\frac{\partial \Phi}{\partial q_i} q_i = 2\left(\frac{\partial \Phi}{\partial q_{(1)}} q_{(1)} + \frac{\partial \Phi}{\partial m_{(1)}} m_{(1)} + \frac{\partial \Phi}{\partial m_{(2)}} m_{(2)}\right),$$

$$\frac{\partial \Phi}{\partial d_{ij}} d_{ij} = \frac{\partial \Phi}{\partial d_{(1)}} d_{(1)} + 2\frac{\partial \Phi}{\partial d_{(2)}} d_{(2)} + 3\frac{\partial \Phi}{\partial d_{(3)}} d_{(3)} + \frac{\partial \Phi}{\partial m_{(1)}} m_{(1)} + 2\frac{\partial \Phi}{\partial m_{(2)}} m_{(2)}. \tag{A.9}$$

3. Let Φ be of the form

$$\Phi(d'_{(2)}, d'_{(3)}). \tag{A.10}$$

According to (A.6) we have

$$\frac{\partial \Phi}{\partial d_{ij}} = \left[\frac{\partial \Phi}{\partial d'_{(2)}}\left(\frac{\partial d'_{(2)}}{\partial d_{(1)}}\delta_{ij} + 2\frac{\partial d'_{(2)}}{\partial d_{(2)}} d_{ij} + 3\frac{\partial d'_{(2)}}{\partial d_{(3)}} d_{ik} d_{kj}\right)\right.$$

$$\left. + \frac{\partial \Phi}{\partial d'_{(3)}}\left(\frac{\partial d'_{(3)}}{\partial d_{(1)}}\delta_{ij} + 2\frac{\partial d'_{(3)}}{\partial d_{(2)}} d_{ij} + 3\frac{\partial d'_{(3)}}{\partial d_{(3)}} d_{ik} d_{kj}\right)\right] \tag{A.11}$$

or, on account of (A.5) and (A.4),

$$\frac{\partial \Phi}{\partial d_{ij}} = 2\frac{\partial \Phi}{\partial d'_{(2)}} d'_{ij} + 3\frac{\partial \Phi}{\partial d'_{(3)}}\left(d'_{ik} d'_{kj} - \frac{1}{3} d'_{(2)} \delta_{ij}\right). \tag{A.12}$$
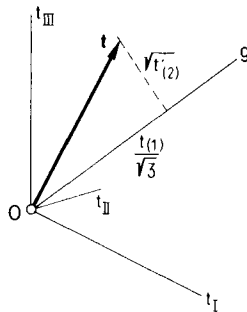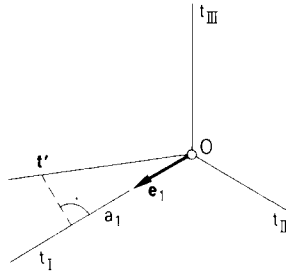

FIG. 13. Vectorial representation **t** of a symmetric tensor t_{ij}.

FIG. 14.   Deviatoric plane $E \perp g$.

It follows that

$$\frac{\partial \Phi}{\partial d_{ij}} d_{ij} = 2 \frac{\partial \Phi}{\partial d'_{(2)}} d'_{(2)} + 3 \frac{\partial \Phi}{\partial d'_{(3)}} d'_{(3)}. \tag{A.13}$$

4. A symmetric tensor t_{ij} with principal values $t_1, \ldots,$ can be represented as a vector $t = (t_1, \ldots)$ in an orthogonal Cartesian coordinate system $t_1,$ $\ldots$ (Fig. 13). The istropic part $t_{(1)}\delta_{ij}/3$ of t_{ij} appears as the projection $t_{(1)}/\sqrt{3}$ of t onto the axis g including equal angles with the positive axes $t_1, \ldots.$ The deviatoric part t'_{ij} is represented by the projection t' of t onto the plane $E \perp g$ passing through the origin O. Its magntiude is

$$|t'| = (t_1'^2 + \cdots)^{1/2} = (t'_{(2)})^{1/2}. \tag{A.14}$$

The unit vector e_1 (Fig. 14) in the projection of the axis t_1 onto the deviatoric plane E is equally inclined with respect to the axes t_{11} and t_{111}. Hence, $e_1 = (2, -1, -1)/\sqrt{6}$, and it follows that the projections $a_1, \ldots,$ of t' onto the projected axes $t_1, \ldots,$ are

$$a_1 = t \cdot e_1 = \sqrt{\tfrac{3}{2}} t_1', \ldots. \tag{A.15}$$

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