UNCUYO
UNIVERSIDAD NACIONAL DE CUYO

FACULTAD DE INGENIERÍA

“VISCOELASTICIDAD”

MATERIALES

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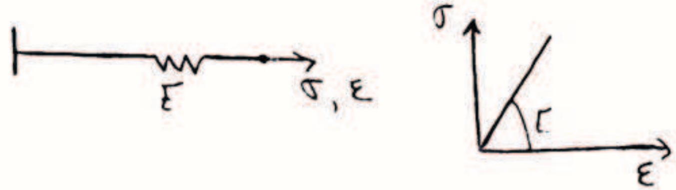
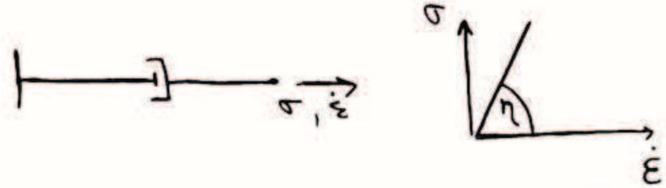


$$\sigma = E\,\varepsilon$$



$$\sigma = \eta\,\dot{\varepsilon}$$

$$\sigma = \eta\,\dot{\varepsilon}$$

$$\Rightarrow \dot{\varepsilon} = \frac{\sigma}{\eta} \Rightarrow \varepsilon(t) = \int_0^t \frac{\sigma}{\eta}\,dt$$

Modelo de Maxwell

$\left.\begin{matrix} E \\ \eta \end{matrix}\right\}$ ctes del material

En el modelo de Maxwell:

$$\boxed{\sigma = \sigma(E) = \sigma(\eta)}$$

La deformación es:

$$\varepsilon = \varepsilon^e + \varepsilon^v \qquad \text{(A)}$$

$$\sigma := E\,\varepsilon^e \Rightarrow \varepsilon^e = \frac{\sigma}{E} \rightarrow \text{Luego: } \dot{\varepsilon}^e = \frac{\dot{\sigma}}{E} \qquad \text{(B)}$$

$$\sigma = \eta\,\dot{\varepsilon}^v \Rightarrow \dot{\varepsilon}^v = \frac{\sigma}{\eta} \qquad \text{(C)}$$

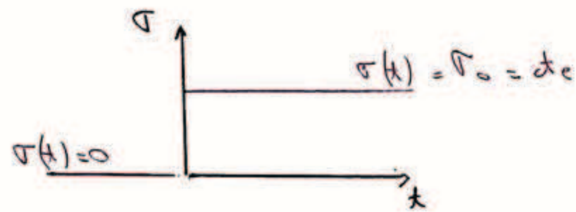
(B) y (C) $\rightarrow$ (A):

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^v$$

$$\Rightarrow \boxed{\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}} \qquad \text{(A')}$$

- 2) Ensayo de creep

$$\boxed{\sigma(t)=\sigma_0} \qquad t>0$$



De la E.D.

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

$$\frac{d\varepsilon}{dt} = \frac{d\varepsilon^e}{dt} + \frac{d\varepsilon^v}{dt} \Rightarrow d\varepsilon = \Bigg(\underbrace{\frac{d\varepsilon^e}{dt}}_{\dot{\varepsilon}^e} + \underbrace{\frac{d\varepsilon^v}{dt}}_{\dot{\varepsilon}^v}\Bigg)\,dt$$

$$\Rightarrow \int d\varepsilon = \int \frac{d\varepsilon^e}{dt}\,dt + \int \frac{d\varepsilon^v}{dt}\,dt$$

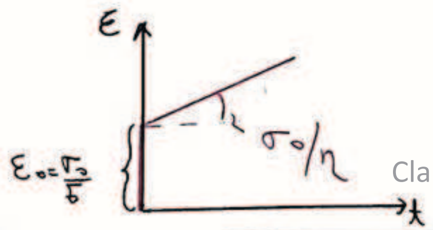
(B); (E)

$$\Rightarrow \varepsilon(t) = \frac{\sigma}{E} + \int \frac{\sigma}{\eta}\,dt$$

$$\Rightarrow \varepsilon(t) = \frac{\sigma_0}{E} + \int_{0^+}^{t} \frac{\sigma_0}{\eta}\,dt \ (0)$$

$$\Rightarrow \boxed{\varepsilon(t) = \overbrace{\frac{\sigma_0}{E}}^{\varepsilon_0} + \frac{\sigma_0}{\eta}\,t}$$

$$= \varepsilon_0 + \frac{\sigma_0}{\eta}\,t$$



La (D) podemos expresarla alternativamente en fc. del tpo característico:

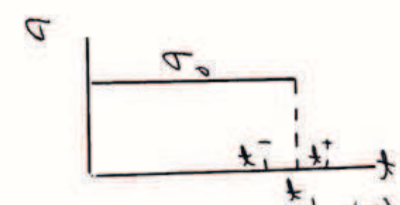
$$\tau = \eta / E \Rightarrow \frac{1}{\tau} = \frac{E}{\eta}$$

$$(E) \quad \frac{\sigma_0}{E} + \frac{\sigma_0}{\eta} t = \frac{\sigma_0}{E}\left(1 + \frac{E}{\eta} t\right) = \underbrace{\frac{\sigma_0}{E}}_{\varepsilon_0}\left(1 + \frac{t}{\tau}\right)$$

$$= \varepsilon_0 + \frac{\varepsilon_0}{\tau} t$$

- Hemos tratado el creep. ¿Qué pasaría para un estado de carga más general? Veamos un ejemplo: una descarga instantánea para un $t = t_1$:

$$\sigma(t) = -\sigma_0 = 0 \qquad t = t_1$$



Solo para amortiguador es cero

$$t = t_1^- \longrightarrow \varepsilon(t) \overset{(E)}{=} \varepsilon(t_1^-) = \frac{\sigma_0}{E} + \frac{\sigma_0}{\eta} t_1$$

$\forall t > t_1^+$

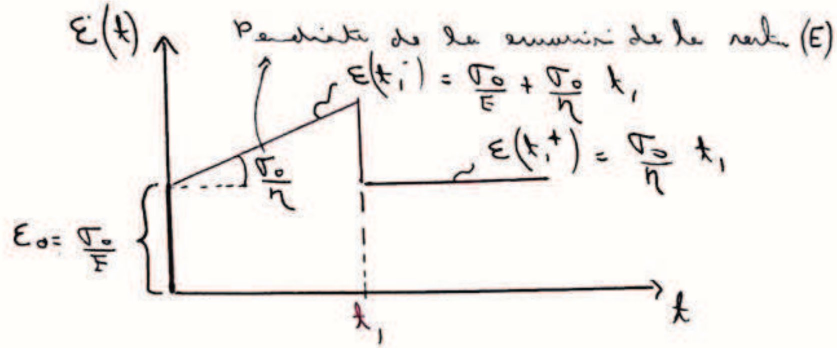
(similar a (D))

$$t = t_1^+ \longrightarrow \sigma_0 = 0$$

$$\varepsilon(t_1^+) = \varepsilon(t_1^-) - \left(\frac{\sigma_0}{E} + \underbrace{\int_{t_1^-}^{t_1^+} \frac{\sigma_0}{\eta}\, dt}_{=0 \leftarrow \sigma(t)=0}\right)$$

$$= \cancel{\frac{\sigma_0}{E}} + \frac{\sigma_0}{\eta} t_1 - \cancel{\frac{\sigma_0}{E}}$$

$$\Rightarrow \boxed{\varepsilon(t_1^+) = \frac{\sigma_0}{\eta} t_1}$$



- Nota: $\exists$ dos magnitudes que se suelen calcular para el problema viscoelástico:

$$\varepsilon(t) = J(t)\,\sigma_0 \quad \Rightarrow \text{se define: } J(t) = \frac{\varepsilon(t)}{\sigma_0}$$

$J(t)$: función de fluencia

$$\sigma(t) = E(t)\,\varepsilon_0$$

: módulo de relajación

se define: $E(t) = \frac{\sigma(t)}{\varepsilon_0}$

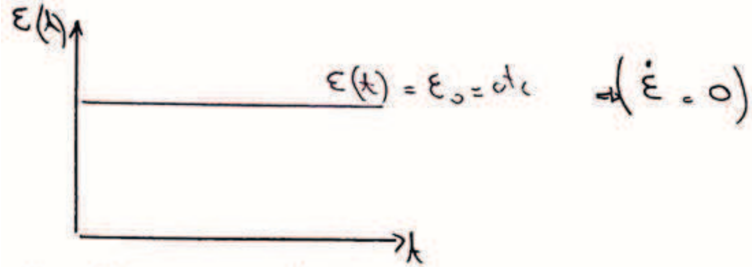
Para el modelo de Maxwell del resultado del ensayo de creep resulta:

Sumando (E) y la deformación

$$J(t) = \frac{1}{E} + \frac{t_1}{\eta}$$

- b) Ensayo de relajación (o de deformación cte)

$\varepsilon(t) = \varepsilon_0 = cte$



La E.D. de (A'):

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

$$\Rightarrow 0 = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

$$\Rightarrow \frac{\dot{\sigma}}{E} = -\frac{\sigma}{\eta}$$

$$\Rightarrow \int \frac{d\sigma}{\sigma} = -\int \frac{E}{\eta}\, dt$$

$$\Rightarrow \ln \sigma = -\frac{E}{\eta} t + cte$$

$$\Rightarrow \sigma = e^{\left(-\frac{E}{\eta} t + cte\right)}$$

$$= e^{-\frac{E}{\eta} t}\, e^{cte}$$ e^{cte} = cte de A

$$= A\, e^{-\frac{E}{\eta} t} \qquad (F)$$

$t = 0 \rightarrow \sigma = A e^0 = A$

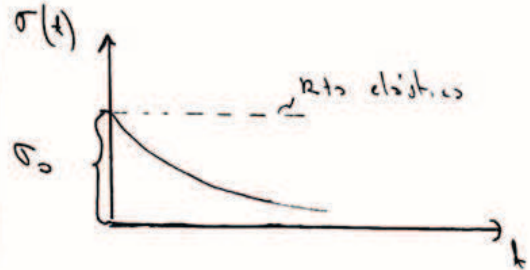
Pero: $\sigma = E\,\varepsilon_0$ en $t = 0$

$\Rightarrow$ Igualando: $A = E\,\varepsilon_0 \qquad (G)$

$\Rightarrow (G) \rightarrow (F)$:

$$\boxed{\sigma = E\,\varepsilon_0\, e^{-\frac{E}{\eta}t}} \qquad (H)$$

$$= \sigma_0\, e^{-\frac{E}{\eta}t}$$



- c) Ensayo con Velocidad de deformación cte

$$\boxed{\dot{\varepsilon} = cte = c}$$

De E.D. (A'):

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} = c \qquad (I)$$

$$(I) \rightarrow \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} = 0$$

$$\Rightarrow \frac{d\sigma}{dt}\frac{1}{E} = -\frac{\sigma}{\eta}$$

$$\Rightarrow \int \frac{d\sigma}{\sigma} = \int -\frac{E}{\eta}\, dt$$

$$\Rightarrow \ln \sigma = -\frac{E}{\eta} t + cte$$

$$\Rightarrow \sigma = e^{\left(-\frac{E}{\eta} t + cte\right)} = e^{-\frac{E}{\eta}t}\, e^{cte} = A\, e^{-\frac{E t}{\eta}} \quad (J)$$

- Solución particular:

$$\frac{\sigma}{\eta} = c = \dot{\varepsilon}$$

$$\Rightarrow \frac{\sigma}{\eta} = \dot{\varepsilon}$$

$$\Rightarrow \sigma = \eta\, \dot{\varepsilon} \qquad (K)$$

- Solución general:

$$\boxed{\sigma(t) = A\, e^{\left(-\frac{E}{\eta}t\right)} + \eta\, \dot{\varepsilon}} \qquad (L)$$

Para $t=0 \rightarrow \sigma(t=0)=0$:

$\Rightarrow (L)$: $0 = A e^{0} + \eta \dot{\varepsilon}$

$$\Rightarrow A = -\eta \dot{\varepsilon} \qquad (M)$$

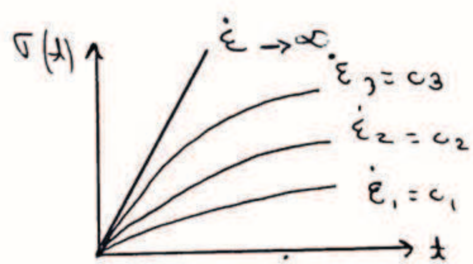
Luego $(M) \rightarrow L$:

$$\boxed{\sigma(t) = -\eta \dot{\varepsilon}\, e^{\left(-\frac{E}{\eta} t\right)} + \eta \dot{\varepsilon}} \qquad (O)$$

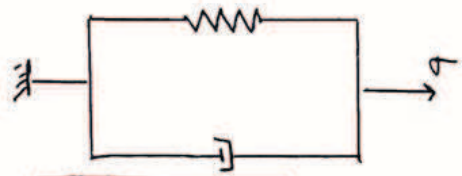
$$= \eta \dot{\varepsilon} \left(1 - e^{\left(-\frac{E}{\eta} t\right)}\right) \qquad (O')$$

$\Downarrow$

De esta ecuación se ve que para $t=0$ la $\sigma=0$:



Modelo de Kelvin-Voigt



$$\boxed{\varepsilon^e = \varepsilon^v = \varepsilon}$$

$$\sigma_{res} = E\,\varepsilon^e = E\varepsilon$$

$$\sigma_{amor.} = \eta\,\dot{\varepsilon}^v = \eta\,\dot{\varepsilon}$$

$$\sigma = \sigma_{resorte} + \sigma_{amortiguador}$$

$$\Rightarrow \boxed{\sigma = E\,\varepsilon + \eta\,\dot{\varepsilon}}$$

De esta última se podría obtener la E.D.

$$\Rightarrow \sigma - E\,\varepsilon = \eta\,\dot{\varepsilon}$$

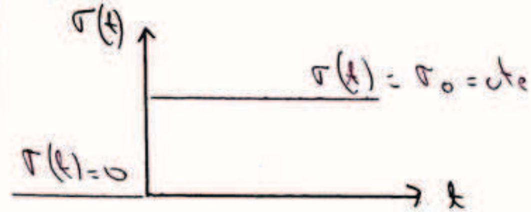
$$\Rightarrow \frac{\sigma}{\eta} - \frac{E}{\eta}\,\varepsilon = \dot{\varepsilon}$$

$$\Rightarrow \boxed{\dot{\varepsilon} + \frac{E}{\eta}\,\varepsilon = \frac{\sigma}{\eta}} \quad (N)$$

(E.D.)

a) Ensayo de creep

$$\boxed{\sigma(t) = \sigma_0} \qquad t > 0$$



De la E.D.:

$$\dot{\varepsilon} + \frac{E}{\eta}\varepsilon = \frac{\sigma_0}{\eta} \qquad (P)$$

• Solución homogénea:

$$(P) \rightarrow \frac{d\varepsilon}{dt} + \frac{E}{\eta}\varepsilon = 0$$

$$\Rightarrow \frac{d\varepsilon}{dt} = -\frac{E}{\eta}\varepsilon$$

$$\Rightarrow \int \frac{d\varepsilon}{\varepsilon} = \int -\frac{E}{\eta}\,dt$$

$$\Rightarrow \ln \varepsilon = -\frac{E}{\eta}t + cte$$

$$\Rightarrow \varepsilon = e^{\left(-\frac{E}{\eta}t + cte\right)} = e^{-\frac{E}{\eta}t}\,\underbrace{e^{cte}}_{=A}$$

$$\Rightarrow \varepsilon = A e^{-\frac{E}{\eta}t} \qquad (Q)$$

• Solución particular:

$$(P) \rightarrow \cancel{\dot{\varepsilon}} + \frac{E}{\eta}\varepsilon = \frac{\sigma_0}{\eta}$$

$$\Rightarrow \varepsilon = \frac{\sigma_0}{\cancel{\eta}} \frac{\cancel{\eta}}{E} \qquad (R)$$

• Solución gral:

De (Q) y (R):

$$\varepsilon = A e^{-\frac{E}{\eta} t} + \frac{\sigma_0}{E} \qquad (S)$$

Para $t = 0 \rightarrow \varepsilon(t=0) = 0$

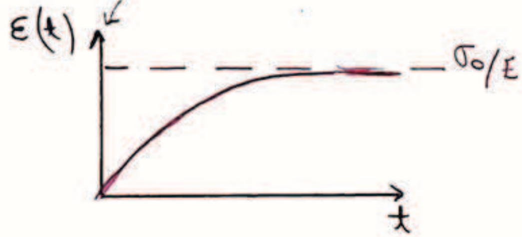
$$\Rightarrow 0 = A e^{0} + \frac{\sigma_0}{E}$$

$$\Rightarrow A = -\frac{\sigma_0}{E} \qquad (T)$$

Luego $(T) \rightarrow (S)$:

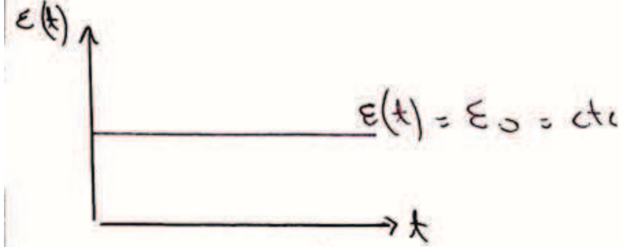
$$\boxed{\varepsilon(t) = -\frac{\sigma_0}{E} e^{-\frac{E}{\eta} t} + \frac{\sigma_0}{E}}$$

$$= \frac{\sigma_0}{E}\left(1 - e^{-\frac{E}{\eta} t}\right)$$



b) Ensayo de relajación:

$\varepsilon(t) = \varepsilon_0 = cte$



$\Rightarrow (\dot{\varepsilon} = 0)$

La E.D. de (N):

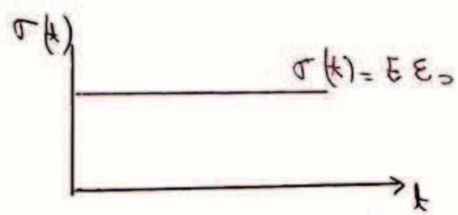
$$\dot{\varepsilon} + \frac{E}{\eta}\,\varepsilon = \frac{\overset{\sigma(t)}{\sigma}}{\eta}$$

$$\Rightarrow \frac{E}{\eta}\,\varepsilon_0 = \frac{\sigma(t)}{\eta}$$

$$\Rightarrow \sigma(t) = \cancel{\eta}\,\frac{E}{\cancel{\eta}}\,\varepsilon_0$$

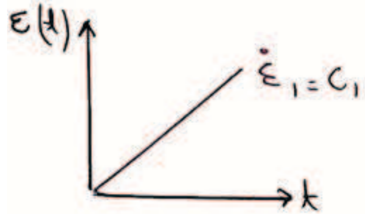
$$\Rightarrow \boxed{\sigma(t) = E\,\overset{cte}{\varepsilon_0}}$$

$= cte$



c) Ensayo con velocidad de deformación cte:

$$\boxed{\dot{\varepsilon} = cte = c}$$



De E.D. (N):

$$\dot{\varepsilon} + \frac{E}{\eta}\varepsilon = \frac{\sigma}{\eta}$$

$$\Rightarrow \sigma = \eta\dot{\varepsilon} + \cancel{\eta}\frac{E}{\cancel{\eta}}\varepsilon$$

$$= \eta\dot{\varepsilon} + E\varepsilon \qquad (U)$$

Luego

$$\dot{\varepsilon} \equiv \frac{d\varepsilon}{dt} \cong \frac{\varepsilon}{t}$$

$$\Rightarrow \varepsilon = \dot{\varepsilon}t \quad (V)$$

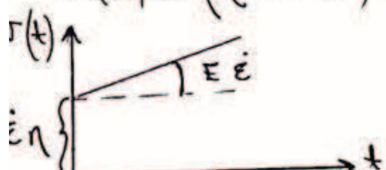
De (V) → (U):

$$\sigma = \eta\dot{\varepsilon} + E\,\dot{\varepsilon}t$$

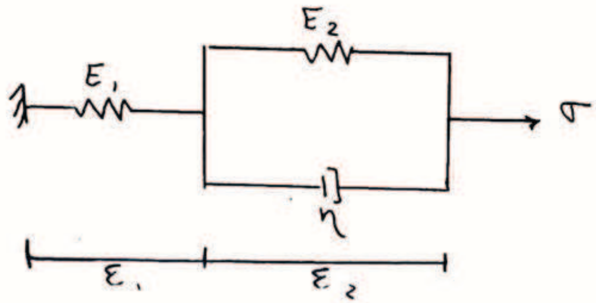
$$\Rightarrow \boxed{\sigma = \dot{\varepsilon}(\eta + Et)} \qquad (X)$$

• Para $t = 0$, considerando (X):

$$\sigma(t=0) = \dot{\varepsilon}(\eta + E \cdot 0) = \underset{cte}{\dot{\varepsilon}}\,\eta$$



Sólido lineal estandar:



Cinemática:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \qquad (1)$$

$$\dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 \qquad (2)$$

De ecuación constitutiva del resorte:

$$\sigma = E_1\,\varepsilon_1 \qquad (3)$$

con lo que:

$$\dot{\sigma} = E_1\,\dot{\varepsilon}_1 \qquad (4)$$

De la E.D. del modelo de Kelvin-V.:

$$(4) \rightarrow \dot{\varepsilon} + \frac{E}{\eta}\varepsilon = \frac{\sigma}{\eta}$$

$$\Rightarrow \sigma = E_2 \varepsilon_2 + \eta \dot{\varepsilon}_2 \qquad (5)$$

Luego de (5):

$$\varepsilon_2 = \frac{1}{E_2}\left(\sigma - \eta \dot{\varepsilon}_2\right) \qquad (6)$$

De (3) y (6) $\rightarrow$ (1):

$$\varepsilon = \frac{\sigma}{E_1} + \frac{1}{E_2}\left(\sigma - \eta \dot{\varepsilon}_2\right) \qquad (7)$$

De (2) $\rightarrow$ (7):

$$\varepsilon = \frac{\sigma}{E_1} + \frac{1}{E_2}\left[\sigma - \eta\left(\dot{\varepsilon} - \dot{\varepsilon}_1\right)\right] \qquad (8)$$

De (4) $\rightarrow$ (8):

$$\varepsilon = \frac{\sigma}{E_1} + \frac{1}{E_2}\left[\sigma - \eta\left(\dot{\varepsilon} - \frac{\dot{\sigma}}{E_1}\right)\right] \qquad (9)$$

$$\Rightarrow \varepsilon = \frac{\sigma}{E_1} + \frac{1}{E_2}\sigma - \frac{1}{E_2}\eta\dot{\varepsilon} + \frac{\eta}{E_2}\frac{\dot{\sigma}}{E_1} \qquad (10)$$

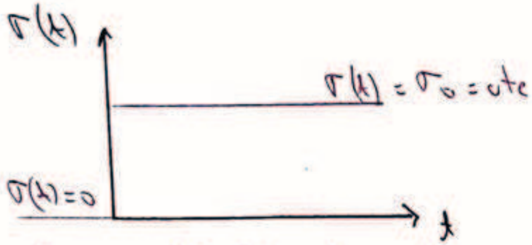
$$\Rightarrow \frac{1}{E_2}\eta\dot{\varepsilon} + \varepsilon = \frac{\eta}{E_1 E_2}\dot{\sigma} + \frac{E_1 + E_2}{E_1 E_2}\sigma \qquad (11)$$

$$\Rightarrow \frac{E_1 \cancel{E_2}}{\cancel{\eta}}\frac{\cancel{\eta}}{\cancel{E_2}}\dot{\varepsilon} + \frac{E_1 E_2}{\eta}\varepsilon = \dot{\sigma} + \frac{E_1 + E_2}{\eta}\sigma \qquad (12)$$

$$\Rightarrow \boxed{E_1\dot{\varepsilon} + \frac{E_1 E_2}{\eta}\varepsilon = \dot{\sigma} + \frac{E_1 + E_2}{\eta}\sigma} \qquad (13)\ \text{E.D.}$$

a) Ensayo de creep:

$$\boxed{\sigma(t) = \sigma_0} \qquad t > 0 \qquad (14)$$



Como $\sigma = \sigma_0 = cte$ entonces:

$$\dot{\sigma} = 0 \qquad (15)$$

De (14)(15) → (13):

$$E_1 \dot{\varepsilon} + \frac{E_1 E_2}{\eta} \varepsilon = \frac{E_1 + E_2}{\eta} \sigma_0 \qquad (16)$$

• Solución homogenea.

$$(16) \rightarrow E_1 \dot{\varepsilon} + \frac{E_1 E_2}{\eta} \varepsilon = 0 \qquad (17)$$

$$\Rightarrow \dot{\varepsilon} + \frac{E_2}{\eta} \varepsilon = 0 \qquad (18)$$

$$\Rightarrow \frac{d\varepsilon}{dt} = -\frac{E_2}{\eta} \varepsilon \qquad (19)$$

$$\Rightarrow \int \frac{d\varepsilon}{\varepsilon} = \int -\frac{E_2}{\eta} dt \qquad (20)$$

$$\Rightarrow \ln \varepsilon = -\frac{E_2}{\eta} t + cte \qquad (21)$$

$$\Rightarrow \varepsilon = e^{\left(-\frac{E_2}{\eta} t + cte\right)} = e^{-\frac{E_2 t}{\eta}} \underbrace{e^{cte}}_{= A \rightarrow \text{otra cte}} \qquad (22)$$

$$\Rightarrow \varepsilon = A e^{-\frac{E_2 t}{\eta}} \qquad (23)$$

• Solución particular

$$(16) \to \cancel{E_1}\dot{\varepsilon} + \frac{E_1 E_2}{\eta}\,\varepsilon = \frac{E_1+E_2}{\eta}\,\sigma_0 \qquad (24)$$

$$\Rightarrow \varepsilon = \frac{\cancel{\eta}}{E_1 E_2}\,\frac{E_1+E_2}{\cancel{\eta}}\,\sigma_0 \qquad (25)$$

• Solución gral:

De (23) y (25):

$$\varepsilon(t) = A\,e^{-\frac{E_2}{\eta}t} + \frac{E_1+E_2}{E_1 E_2}\,\sigma_0 \qquad (26)$$

Para $t=0$:

$$(3) \to \varepsilon_0 = \frac{\sigma_0}{E_1} \qquad (27)$$

$$\Rightarrow (27) \to (26): \frac{\sigma_0}{E_1} = A\,\underbrace{e^{-\frac{E_2}{\eta}t=0}}_{=1} + \frac{E_1+E_2}{E_1 E_2}\,\sigma_0 \qquad (28)$$

$$\Rightarrow A = \frac{\sigma_0}{E_1} - \frac{E_1+E_2}{E_1 E_2}\,\sigma_0 \qquad (29)$$

$$= \sigma_0\left(\frac{1}{E_1} - \frac{E_1+E_2}{E_1 E_2}\right) = \frac{\sigma_0}{E_1}\left(1 - \frac{E_1+E_2}{E_2}\right) \qquad (30)$$

$$= \frac{\sigma_0}{E_1}\left(\frac{\cancel{E_2} - E_1 - \cancel{E_2}}{E_2}\right) = \frac{\sigma_0}{\cancel{E_1}}\left(-\frac{\cancel{E_1}}{E_2}\right) \qquad (31)$$

$$= -\frac{\sigma_0}{E_2} \qquad (32)$$

Luego:

$$(32) \to (26): \quad \varepsilon(t) = -\frac{\sigma_0}{E_2}\,e^{-\frac{E_2}{\eta}t} + \frac{E_1+E_2}{E_1 E_2}\,\sigma_0 \qquad (33)$$

$$\Rightarrow \boxed{\varepsilon(t) = \frac{\sigma_0}{E_2}\left(\frac{E_1+E_2}{E_1} - e^{-\frac{E_2}{\eta}t}\right)} \qquad (34)$$

- Veamos ahora gráficamente como es la $\varepsilon(t)$:

. Para $t \to \infty$:

$$(34) \to : \varepsilon(t \to \infty) = \frac{\sigma_0}{E_2}\,\frac{E_1+E_2}{E_1} \qquad (35)$$

$$= \frac{\sigma_0}{\dfrac{E_1 E_2}{E_1+E_2}} \qquad (36)$$

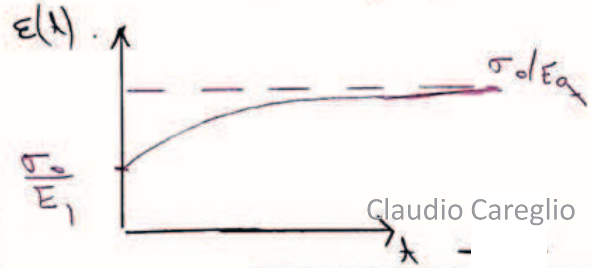
$$= \frac{\sigma_0}{E_q}. \qquad (37)$$

. Para $t = 0$:

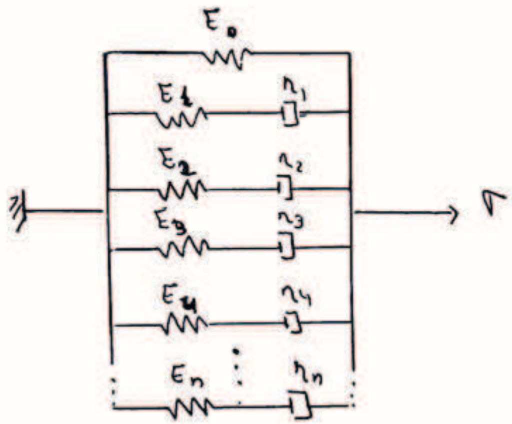
$$(34) \to \varepsilon(t=0) = \frac{\sigma_0}{E_2}\left(\frac{E_1+E_2}{E_1} - e^{0}\right) \qquad (38)$$

$$= \frac{\sigma_0}{\cancel{E_2}}\,\frac{\cancel{E_2}}{E_1} = \frac{\sigma_0}{E_1} \qquad (39)$$

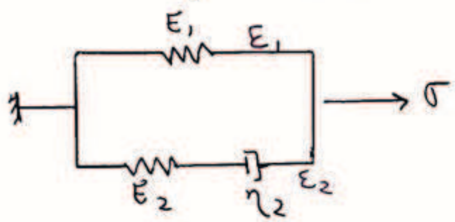
. Por lo que el gráfico de $\varepsilon(t)$:



Modelo de Maxwell generalizado



Para nuestro análisis consideremos como ejemplo un modelo de Maxwell generalizado de 1 rama.



Cinemática,

$$\varepsilon = \varepsilon_1 = \varepsilon_2 \qquad [1]$$

Por equilibrio :

$$\sigma = \sigma_1 + \sigma_2 \qquad [2]$$

$$\Rightarrow \dot{\sigma} = \dot{\sigma}_1 + \dot{\sigma}_2 \qquad [3]$$

De ecuación constitutiva del resorte en la rama 1:

$$\sigma_1 = E_1 \varepsilon_1 \qquad [4]$$

$$\overset{[1]}{=} E_1 \varepsilon \qquad [5]$$

$$\Rightarrow \dot{\sigma}_1 = E_1 \dot{\varepsilon} \qquad [6]$$

En la rama del modelo de Maxwell:

$$\varepsilon_2 = \varepsilon_{2\,amortiguador} + \varepsilon_{2\,resorte} \qquad [7]$$

$$\overset{[1]}{=} \varepsilon \qquad [8]$$

$$\Rightarrow \dot{\varepsilon}_2 = \dot{\varepsilon} \qquad [9]$$

$$= \dot{\varepsilon}_{2\,amort.} + \dot{\varepsilon}_{2\,resorte} \qquad [10]$$

$$\sigma_{2\,resor.} = E_2\, \varepsilon_{2\,resort.} \qquad [11]$$

$$= \sigma_2 \qquad [12]$$

$$\Rightarrow \dot{\varepsilon}_{2\,res.} = \frac{\dot{\sigma}_2}{E_2} \qquad [13]$$

$$\sigma_{2\,amort.} = \eta_2\, \dot{\varepsilon}_{2\,amort.} \qquad [14]$$

$$= \sigma_2 \qquad [15]$$

$$\Rightarrow \dot{\varepsilon}_{2\,amort.} = \frac{\sigma_2}{\eta_2} \qquad [16]$$

de [13] y [16] $\rightarrow$ [10]:

$$\dot{\varepsilon}_2 = \dot{\varepsilon} = \frac{\sigma_2}{\eta_2} + \frac{\dot{\sigma}_2}{E_2} \qquad [17]$$

$$\Rightarrow \dot{\varepsilon} - \frac{\sigma_2}{\eta_2} = \frac{\dot{\sigma}_2}{E_2} \qquad [18]$$

$$\Rightarrow \dot{\sigma}_2 = E_2\left(\dot{\varepsilon} - \frac{\sigma_2}{\eta_2}\right) \qquad [19]$$

De [6] y [19] $\rightarrow$ [3]:

$$\dot{\sigma} = E_1\,\dot{\varepsilon} + E_2\left(\dot{\varepsilon} - \frac{\sigma_2}{\eta_2}\right) \qquad [20]$$

$$= E_1\,\dot{\varepsilon} + E_2\,\dot{\varepsilon}\ \frac{E_2}{\eta_2} \ldots \qquad [21]$$

pero:

$[2] \rightarrow \sigma_2 = \sigma - \sigma_1$ [22]

$= \sigma - E_1 \varepsilon$ [23]

$[23] \rightarrow [21]$:

$$\dot{\sigma} = E_1 \dot{\varepsilon} + E_2 \dot{\varepsilon} - \frac{E_2}{\eta_2}\left(\sigma - E_1 \varepsilon\right) \quad [24]$$

$$= \left(E_1 + E_2\right) \dot{\varepsilon} - \frac{E_2}{\eta_2}\sigma + \frac{E_2}{\eta_2}E_1 \varepsilon \quad [25]$$

$$\Rightarrow \boxed{\dot{\sigma} + \frac{E_2}{\eta_2}\sigma = \frac{E_2 E_1}{\eta_2} \varepsilon + \left(E_1 + E_2\right) \dot{\varepsilon}} \quad \text{E.D.} [26]$$

lo que puede reescribirse:

$$\dot{\sigma} + \underbrace{p_0}_{=\frac{E_2}{\eta_2}} \sigma = \underbrace{q_0}_{=\frac{E_1 E_2}{\eta_2}} \varepsilon + \underbrace{q_1}_{=E_1+E_2} \dot{\varepsilon}$$

p_0, q_0, q_1: coeficientes

La ecuación [26] se puede utilizar para analizar creep, relajación, etc.

a) Ensayo de creep: (Ejercicio para alumnos)

b) Ensayo de relajación: (Ejercicio para alumnos)

Comportamiento en equilibrio:

Dado que Maxwell generalizado suele usarse en la práctica es que se analiza más en detalle.

Con este fin se estudia el comportamiento en equilibrio y en transitorio (no equilibrio).

Para un tiempo grande $t \to \infty$ se alcanza un estado de equilibrio y el modelo reológico debe estar en reposo. En este caso:

$$\dot{\varepsilon} = 0$$

$$\Rightarrow \sigma_{2\,amort.} = 0$$

por equilibrio resulta:

$$\sigma_{2\,resorte} = 0$$

$$\sigma = \underbrace{\sigma_{\infty}}_{\text{Tensión en la rama con resorte solo}} + \underbrace{\sigma_{n.eq.}}_{\text{Tensión que corresponde a un estado de no equilibrio}}$$

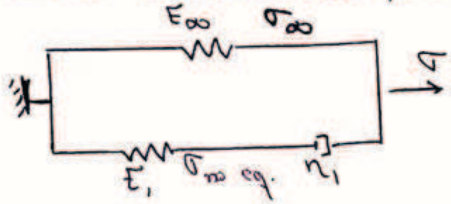
Tensión en la rama con resorte solo

↓

y al módulo de Young se lo llama E_{∞}

Tensión que corresponde a un estado de no equilibrio (tiene lugar en c/u de las ramas de Maxwell)

- Planteo termodinámico y variables internas



Cinemática:

$\varepsilon = \varepsilon_1$

$\varepsilon_1 = \varepsilon_1^e + \varepsilon_1^v$

$\Rightarrow \dot{\varepsilon}_1 = \dot{\varepsilon}_1^e + \dot{\varepsilon}_1^v$

Por equilibrio:

$\sigma = \sigma_\infty + \sigma_{no\ eq.}$ A

De ecuación constitutiva:

$\sigma_\infty = E_\infty \varepsilon$ B

$\sigma_{no\ eq.} = E_1 \varepsilon_1^e$

$= E_1 (\varepsilon - \varepsilon_1^v)$

$= \eta_1 \dot{\varepsilon}_1^v$ C

Se puede obtener una ecuación de la cual se obtenga la evolución de ε_1^v

$$\boxed{\dot{\varepsilon}_1^v = \frac{E_1}{\eta_1}\left(\varepsilon - \varepsilon_1\right) = \frac{1}{\tau}\left(\varepsilon - \varepsilon_1\right)}$$

Partiendo de las expresiones de la energía libre (asumimos formas cuadráticas para ambos resortes):

$$\psi = \psi_\infty + \psi_{no\ eq.}$$

$$= \underbrace{\frac{1}{2}\,\varepsilon\, E_\infty\, \varepsilon}_{\boxed{E}} + \underbrace{\frac{1}{2}\left(\varepsilon - \varepsilon_1^v\right) E_1 \left(\varepsilon - \varepsilon_1^v\right)}_{\boxed{F}} \quad \boxed{D}$$

Luego:

$$\dot{\psi} = \dot{\psi}_\infty + \dot{\psi}_{no\ eq.}$$

$$= \frac{\partial \psi_\infty}{\partial \varepsilon}\,\dot{\varepsilon} + \frac{\partial \psi_{no\ eq.}}{\partial\left(\varepsilon - \varepsilon_1^v\right)}\left(\dot{\varepsilon} - \dot{\varepsilon}_1^v\right)$$

Considerando la desigualdad de Clausius:

$$-\dot{\psi} + \sigma : \dot{\varepsilon} \geq 0$$

$$\Rightarrow -\left(\frac{\partial \psi_\infty}{\partial \varepsilon}\,\dot{\varepsilon} + \frac{\partial \psi_{no\ eq.}}{\partial\left(\varepsilon - \varepsilon_1^v\right)}\left(\dot{\varepsilon} - \dot{\varepsilon}_1^v\right)\right) + \sigma : \dot{\varepsilon} \geq 0$$

$$\Rightarrow -\frac{\partial \psi_\infty}{\partial \varepsilon}\,\dot{\varepsilon} - \frac{\partial \psi_{no\ eq.}}{\partial\left(\varepsilon - \varepsilon_1^v\right)}\,\dot{\varepsilon} + \frac{\partial \psi_{no\ eq.}}{\partial\left(\varepsilon - \varepsilon_1^v\right)}\,\dot{\varepsilon}_1^v + \sigma : \dot{\varepsilon} \geq 0$$

reagrupando términos:

$$\left(-\frac{\partial \psi_\infty}{\partial \varepsilon} + \sigma_\infty\right)\dot{\varepsilon} + \left(-\frac{\partial \psi_{no\ eq.}}{\partial\left(\varepsilon - \varepsilon_1^v\right)} + \sigma_{no\ eq.}\right)\dot{\varepsilon} +$$

$$+ \frac{\partial \psi_{no\ eq.}}{\partial\left(\varepsilon - \varepsilon_1^v\right)}\,\dot{\varepsilon}_1^v \geq 0$$

Analizando ahora c/términos

$$\sigma_\infty = \frac{\partial \psi_\infty}{\partial \varepsilon}$$

$$\overset{\boxed{E}}{=} \frac{\partial}{\partial \varepsilon}\left[\frac{1}{2}\,\varepsilon\, E_\infty\, \varepsilon\right]$$

$$= E_\infty\, \varepsilon$$

$$\sigma_{no\ eq.} = \frac{\partial \psi_{no\ eq.}}{\partial(\varepsilon-\varepsilon_1^v)} \overset{\boxed{E}}{=} \frac{\partial}{\partial(\varepsilon-\varepsilon_1^v)}\left[\frac{1}{2}\left(\varepsilon-\varepsilon_1^v\right)E_1\left(\varepsilon-\varepsilon_1^v\right)\right]$$

$$= E_1\left(\varepsilon-\varepsilon_1^v\right) \quad \boxed{G}$$

$$\overset{\boxed{C}}{=} \eta_1\,\dot{\varepsilon}_1^v \quad \boxed{H}$$

$$D = \frac{\partial \psi_{no\ eq}}{\partial(\varepsilon-\varepsilon_1^v)}\,\dot{\varepsilon}_1^v \geq 0$$

$$\Rightarrow D \overset{\boxed{G}}{=} E_1\left(\varepsilon-\varepsilon_1^v\right)\dot{\varepsilon}_1^v \quad \boxed{I}$$

$$\overset{\boxed{H}}{=} \eta_1\,\dot{\varepsilon}_1^v\,\dot{\varepsilon}_1^v$$

$$\Rightarrow D = \eta_1\,\dot{\varepsilon}_1^{v\,2} \geq 0 \quad \boxed{J}$$

De $\boxed{J}$:

$$E_1\left(\varepsilon-\varepsilon_1^v\right)\cancel{\dot{\varepsilon}_1^v} = \eta_1\,\dot{\varepsilon}_1^v$$

$$\Rightarrow \boxed{\begin{aligned}\dot{\varepsilon}_1^v &= \frac{E_1}{\eta_1}\left(\varepsilon-\varepsilon_1\right)\\ &= \frac{1}{\tau_1}\left(\varepsilon-\varepsilon_1\right)\end{aligned}}$$

Ec. de evolución $\boxed{K}$

$$p = -\frac{\partial \psi}{\partial \varepsilon_1^v}$$

$$\Rightarrow p \overset{\boxed{D}}{=} -\frac{\partial}{\partial \varepsilon_1^v}\left[\frac{1}{2}\,\varepsilon\, E_\infty\, \varepsilon + \frac{1}{2}\left(\varepsilon - \varepsilon_1^v\right) E_1 \left(\varepsilon - \varepsilon_1^v\right)\right]$$

$$= -\left[\left(\varepsilon - \varepsilon_1^v\right)(-1)E_1\right]$$

$$= +E_1\left(\varepsilon - \varepsilon_1^v\right) \qquad \boxed{L}$$

Comparando $\boxed{L}$ con $\boxed{B}$:

$P = \sigma_{\text{no eq.}}$

Se puede verificar que:

$$D = p \cdot \dot{\varepsilon}_1^v = E_1\left(\varepsilon - \varepsilon_1^v\right)\dot{\varepsilon}_1^v$$

$$= \eta_1\, \dot{\varepsilon}_1^{v\,2}$$