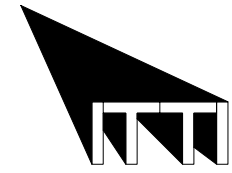

ITTI Finite Element Training Project

Static I : Plate & Shell Analysis

Developed by

Dr Rae S Gordon

MCC UMIST Support Unit

For further information on the booklet please do contact:

OR

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Contents

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Notation

A	Element Area
E	Youngs modulus of elasticity
G	modulus of rigidity
$\underline{i}$	unit vector in *x* direction
$\underline{j}$	unit vector in *y* direction
$\underline{k}$	unit vector in *z* direction
k	stiffness component
$L_{1,etc}$	area coordinate
$L_{x,etc}$	length dimension
N	shape function
P	nodal load component
q	distributed load
r	radial cylindrical polar coordinate
t	element thickness
u	displacement component in *x* direction
U	strain energy
v	displacement component in *y* direction
w	displacement component in *z* direction
W	work done by external loads
x	cartesian coordinate
y	cartesian coordinate
z	cartesian coordinate / axial cylindrical polar coordinate
{a}	displacement vector
[B]	strain shape function matrix
[C]	direction cosine matrix
[D]	elasticity matrix
{f}	nodal force vector
[J]	Jacobian matrix
[K]	stiffness matrix
[N]	shape function matrix
α	coefficient of assumed solution polynomial
γ	shear strain component
ϵ	direct strain components

ζ	intrinsic coordinate
η	intrinsic coordinate
θ	cylindrical polar coordinate
ν	Poissons ratio
ξ	intrinsic coordinate
Π	total potential energy
σ	direct stress component
τ	shear stress component
$\{\epsilon\}$	strain vector
$\{\sigma\}$	stress vector

1 Plate Bending

1.1 Introduction

Much of the pioneering work carried out developing the finite element method was done by the aerospace industry. The stress integrity of the components and assemblies in this industry is of paramount concern, for safety reasons. As an example, the wings of an aircraft consists of a series of spars at various positions along the length of the wing and these are covered by a series of thin plates. This form of structure has a very good strength to weight ratio. The spars give the structure strength and the use of the plates allows a reduced weight over a solid wing. This structure is very complex and difficult to analyse without the use of a tool such as the finite element method. Therefore, the development of a range of elements was required. One area of the finite element method that has received a great deal of attention from researchers is the analysis of plate bending. These elements are two-dimensional, and generally have either five or six degrees of freedom per node. This makes these elements expensive to use as a single second order quadrilateral plate element has 48 degrees of freedom.

It could be argued that plate type structures, being three-dimensional could be analysed using three-dimensional brick elements. However, because the thickness of the plate is much smaller than the other dimensions, aspect ratio error would be induced in the analysis resulting in poor accuracy of results. On the other hand, if the three-dimensional element were reduced in size so that the other dimensions of the element were comparable with the thickness, the error due to the aspect ratio would be eliminated. However, the number of elements required to achieve this, would result in a very large model which would require a massive amount of computer resources to run.

There are a number of criteria, which a three-dimensional structure must satisfy to be classed as a plate. These criterion are as follows,

1. The thickness of the plate is small compared with the other dimensions of the plate.

2. The mid-surface of the structure is a neutral surface during bending.

3. Out-of-plane loading can be applied, (in-plane loading can be included if a two-dimensional finite plane stress element is included in the derivation of the plate bending element).

4. The out-of-plane deflections are small compared with the thickness of the plate.

5. The stress normal to the mid-surface is negligible compared with the bending stresses.

It has been previously stated that the derivation and development of plate bending elements has long been a favourite topic of researchers and many different plate elements exist. Therefore a number of additional assumption may be made for these theories. The reason that these elements receive so much attention is that there are different theories depending on the thickness of the plate. The continuity of slope along edges and at the corners of thin plates cannot be maintained within an element and therefore cannot be maintained across element boundaries.

In this text the most basic, thin and thick plate element theories are used to introduce the topic of plate bending elements, but the reader should be aware that there are many variations of plate elements. In most finite element packages there is no distinction made between plate and shell elements and generally only shell elements will be given. The reason for this is that the shell element is an extension of the plate element. The problem of having to decide whether a thick or thin shell element should be selected is eliminated when using most commercial finite element codes. The programs either use a more elaborate shell element which can be used for both thick and thin plates or makes the choice based on the model data.

The difference between a plate element and a shell element is that the plate element is flat and lies on a single plane. A shell is a structure that can be derived from a plate by initially forming the mid-plane to a singly or doubly curved surface. A shell element must take in-plane as well as out-of-plane loading into account.

2 Thin Plate Elements

2.1 Introduction

The derivation of the analysis of thin plates is based on the classical plate theory developed by Kirchhoff. There are a number of basic criteria that must be satisfied to allow a three-dimensional structure to be defined as a plate. These criterion are listed in the introduction to this booklet. In the derivation of thin plates Kirchhoff made some additional assumptions. These are,

1. The normal to the mid-plane before deformation, remains normal to the mid-plane after deformation.

2. The shear stress normal to the mid-plane is negligible.

The first assumption basically ignores any effects due to shear deformation. In thin plates this is a reasonable assumption to make as the thickness is so small, it is not so reasonable for thicker plates.

2.2 Kirchhoff Plate Element

Consider the plate shown below in *Figure 1*, of thickness *t*. The coordinate system is defined such that the *x* and *y* axes are coincident with the neutral surface and the *z* axis is positive in the direction of the bottom of the plate.

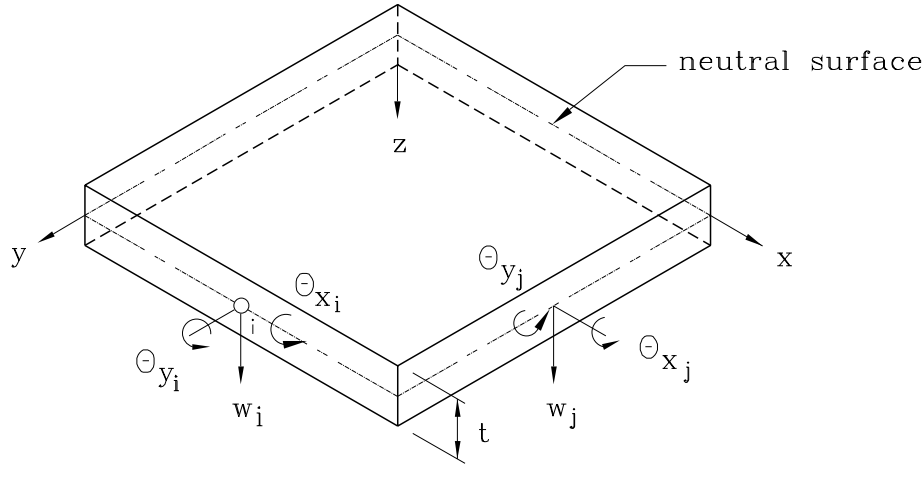

Figure 1 : A plate with thickness, t.

All nodes, such as *i* and *j*, in *Figure 1*, lie on the neutral surface. Each node has three degrees of freedom, two rotations about the in-plane axes and a displacement in the direction of the positive *z* axis.

The finite element equations can be obtained using the theory of minimum total potential energy, using similar procedures to those employed in the derivation of the two and three-dimensional finite element analysis. The total potential energy is given by the difference between the strain energy of the structure and the work done by the external loads.

The work done is given by the load multiplied by the displacement (or moment times the rotation). For a single node this can be written as,

$$W = \begin{bmatrix} w & \theta_x & \theta_y \end{bmatrix} \begin{Bmatrix} P_z \\ M_x \\ M_y \end{Bmatrix} \tag{1}$$

The strain energy of any structure is given by,

$$U = \int_{Vol} \frac{\{\sigma\}^t \{\epsilon\}}{2} dVol \tag{2}$$

In this case due to the basic assumptions made for a plate, the components of stress and strain are,

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad \{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{3}$$

The stress is related to the strain by the generalised Hook's law, for the direct strains this is reduced to,

$$\begin{aligned} \epsilon_x &= \sigma_x - nu\,\sigma_y \\ \epsilon_y &= \sigma_y - nu\,\sigma_x \end{aligned} \tag{4}$$

and the shear stress / shear strain relationship is,

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{5}$$

where,

$$G = \frac{E}{2(1-nu)} \tag{6}$$

Equations (4) and (5) can be written in matrix form and re-arranged to give,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-nu^2} \begin{bmatrix} 1 & nu & 0 \\ nu & 1 & 0 \\ 0 & 0 & \frac{1-nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{7}$$

This is identical to the relationship obtained for the two-dimensional plane stress analysis. Equation (7) is generally written as,

$$\{\sigma\} = [D]\{\epsilon\} \tag{8}$$

Using equation (8) the strain energy equation (2) can be written as,

$$U = \frac{1}{2}\int_x\int_y\int_z \{\epsilon\}^t [D]\{\epsilon\}\, dzdydx \tag{9}$$

The strain-displacement relationship for small displacement theory is,

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned} \tag{10}$$

It can be deduced that the strain components will be zero on the neutral surface as the in-plane displacements will be zero. If *Figure 1*, is viewed in such a way that the surface with node *i* is viewed directly, as shown in *Figure 2*, that there is a displacement in the direction of the *x* axis at points throughout the section other than the neutral surface.

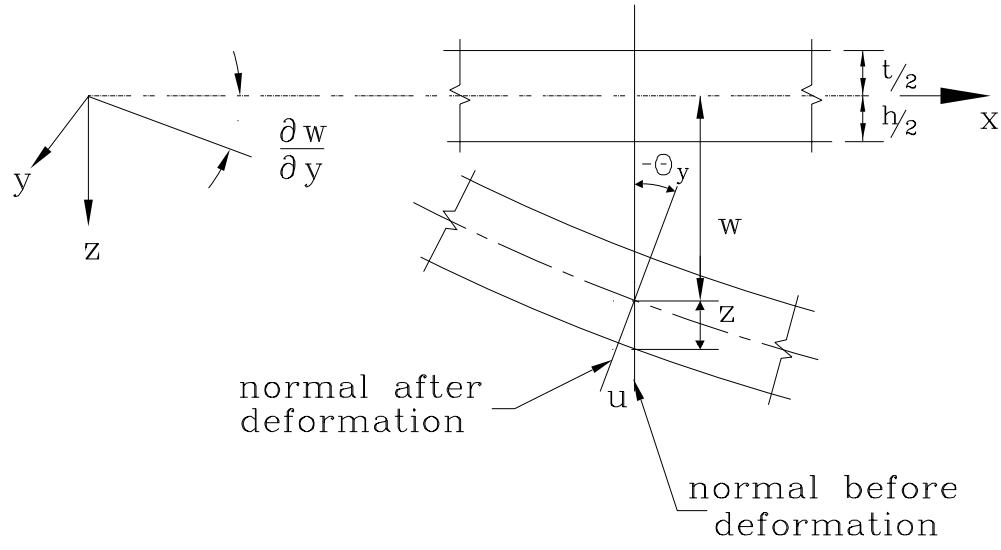

Figure 2 : Plate viewed in the direction of y axis.

Note that the normal to the mid-surface before deformation remains normal to the mid-surface after deformation. The rotation (slope) after deformation is θ_y which is in a clockwise direction and hence, due to the sign convention adopted, is negative. Examining the angles in *Figure 2*, it can be seen that,

$$\frac{\partial w}{\partial x} = -\theta_y \tag{11}$$

It can also be seen that,

$$u = z\theta_y \tag{12}$$

(Noting that as shown in *Figure 2*, u is a negative displacement and also θ_y is a negative rotation).

Substituting equation (11) into equation (12) gives,

$$u = -z\frac{\partial w}{\partial x} \tag{13}$$

Figure 1 can also be viewed in such a way that node *j* is directly in front of the viewer. This is shown below in *Figure 3*.

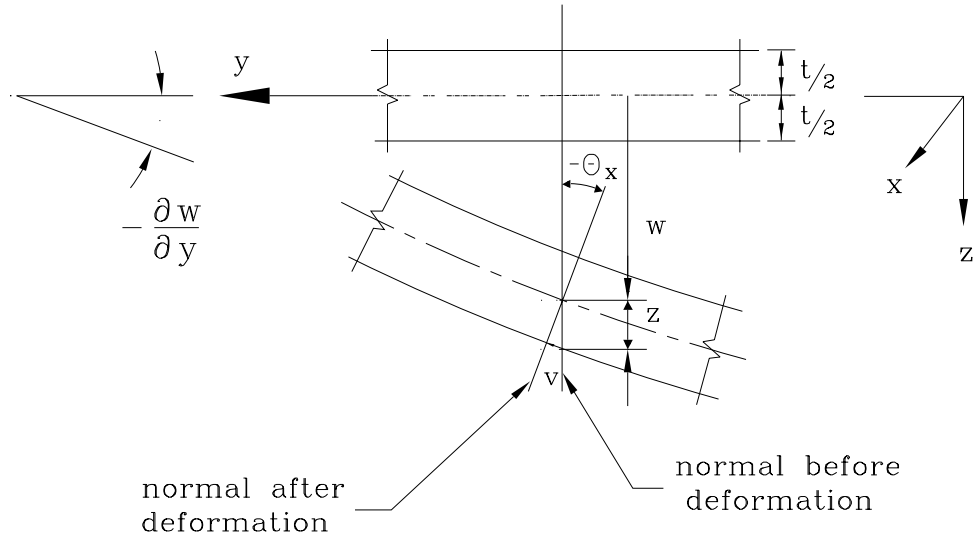

Figure 3 : Plate viewed in the direction of the y axis.

The rotation θ_x is clockwise and is hence negative. The slope of the tangent from the deflected neutral surface to the axis also has a negative slope. Examining the angles in *Figure 3*, it can be seen that,

$$\frac{\partial w}{\partial y} = \theta_x \tag{14}$$

It can also be seen that,

$$v = -z\,\theta_x \tag{15}$$

Substituting equation (14) into equation (15) gives,

$$v = -z\frac{\partial w}{\partial y} \tag{16}$$

Using equations (13) and (16) the strain-displacement relationships can be re-written as,

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = -z\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = -z\frac{\partial^2 w}{\partial x^2} \\
\varepsilon_y &= \frac{\partial v}{\partial y} = -z\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = -z\frac{\partial^2 w}{\partial y^2} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\left\{\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\right\} = -2z\frac{\partial^2 w}{\partial x \partial y}
\end{aligned} \tag{17}$$

It can be seen from equations (11), (13), (14) and (16) that all the components are functions of the single displacement term, *w*. This is known as ***irreducible*** as it is dependant on only one variable.

As with other finite element derivations an element is defined and a polynomial distribution of the displacement function is assumed. In this case, however, only the displacement *w*, at any point is required in terms of the nodal parameters, as the other degrees of freedom can be calculated from the displacement, *w*. The displacement function can be written in terms of the nodal degrees of freedom and shape functions as,

$$w = \sum_{i=1}^{n} \left(a_i w_i + b_i \frac{\partial w_i}{\partial x} + c_i \frac{\partial w_i}{\partial y} \right) \tag{18}$$

where a_i, b_i and c_i are Hermitian shape functions. The element shape functions are discussed in more detail in subsequent sections, as they are not as straight forward as other two-dimensional elements. Problems arise with the continuity of the slopes within an element, never mind between elements.

Using the relationships given in equations (11) and (14) equation (18) can be written as,

$$w = \sum_{i=1}^{n} \left(a_i w_i - b_i \theta_{y_i} + c_i \theta_{x_i} \right) \tag{19}$$

or in matrix form as,

$$w = \sum_{i=1}^{n} \{a\}^t \{N\} \tag{20}$$

where,

$$\{a\} = \begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} \qquad \{N\} = \begin{Bmatrix} a_i \\ c_i \\ -b_i \end{Bmatrix} \tag{21}$$

Equation (20) allows the strains at any point to be defined in terms of the nodal degrees of freedom and the shape functions, thus,

$$\begin{aligned} \epsilon_x &= -z \sum_{i=1}^{n} \{a\}^t \frac{\partial^2}{\partial x^2} \{N\} \\ \epsilon_y &= -z \sum_{i=1}^{n} \{a\}^t \frac{\partial^2}{\partial y^2} \{N\} \\ \gamma_{xy} &= -2z \sum_{i=1}^{n} \{a\}^t \frac{\partial^2}{\partial x \partial y} \{N\} \end{aligned} \tag{22}$$

These strain components can be written in matrix form as,

$$\{\epsilon\} = [B]\{a\} \tag{23}$$

where,

$$[B] = -z\begin{bmatrix} \frac{\partial^2 N_1}{\partial x^2} & \ldots\ldots & \frac{\partial^2 N_n}{\partial x^2} \\ \frac{\partial^2 N_1}{\partial y^2} & \ldots\ldots & \frac{\partial^2 N_n}{\partial y^2} \\ 2\frac{\partial^2 N_1}{\partial x \partial y} & \ldots\ldots & 2\frac{\partial^2 N_n}{\partial x \partial y} \end{bmatrix} \tag{24}$$

Using equation (23), the strain energy equation (9) can be written as,

$$U = \frac{1}{2}\{a\}\left[\int_x\int_y\int_z [B]^t[D][B]\,dz\,dy\,dx\right]\{a\} \tag{25}$$

The total potential energy becomes,

$$\Pi = \frac{1}{2}\{a\}\left[\int_x\int_y\int_z [B]^t[D][B]\,dz\,dy\,dx\right]\{a\} - \{a\}^t\{f\} \tag{26}$$

The finite element equations are obtained when the potential energy is a minimum (i.e. when the loaded structure attains an equilibrium position). For minimum potential energy,

$$\frac{\partial \Pi}{\partial \{a\}^t} = 0 \tag{27}$$

This gives,

$$\left[\int_x\int_y\int_z [B]^t[D][B]\,dz\,dy\,dx\right]\{a\} = \{f\} \tag{28}$$

where,

$$[K] = \left[\int_x\int_y\int_z [B]^t[D][B]\,dz\,dy\,dx\right] \tag{29}$$

is the element stiffness matrix. The function being integrated in the stiffness matrix, through the thickness of the plate is a function of z^2 only. Therefore,

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} z^2[D]\,dz = \frac{t^3}{12}[D] \tag{30}$$

The [D] matrix is modified to,

$$[D'] = \frac{Et^3}{12(1-nu^2)}\begin{bmatrix} 1 & nu & 0 \\ nu & 1 & 0 \\ 0 & 0 & \frac{1-nu}{2} \end{bmatrix} \tag{31}$$

Thus the stiffness matrix becomes,

$$[K] = \left[\int_x\int_y [B]^t [D'] [B]\, dy\, dx\right] \tag{32}$$

The shape functions for the Kirchhoff element can be evaluated in terms of intrinsic coordinates allowing an intrinsic element to be defined. The stiffness matrix can be evaluated using numerical integration techniques, and this aid the numerical implementation of the process.

2.3 Shape Function for Kirchhoff Plate Elements

In the previous section, the derivation for the analysis of Kirchhoff plate elements was given, but it was slightly vague on the subject of element shape functions. This was deliberate as there are a number of difficulties associated with the shape functions for thin plate elements. This is an area where a great deal of time and effort has been spent by researchers.

The plate element requires not only continuity of the displacement but of the slope of the element at it's interface. The slope being the first derivative of the displacement. This is known as $\boldsymbol{C_1}$ continuity. This is impossible to achieve. If a four node rectangular element is considered, as shown in *Figure 4*, a displacement function can be written along a particular side of the element, say from node 1 to node 2, such that,

$$w = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + ... \tag{33}$$

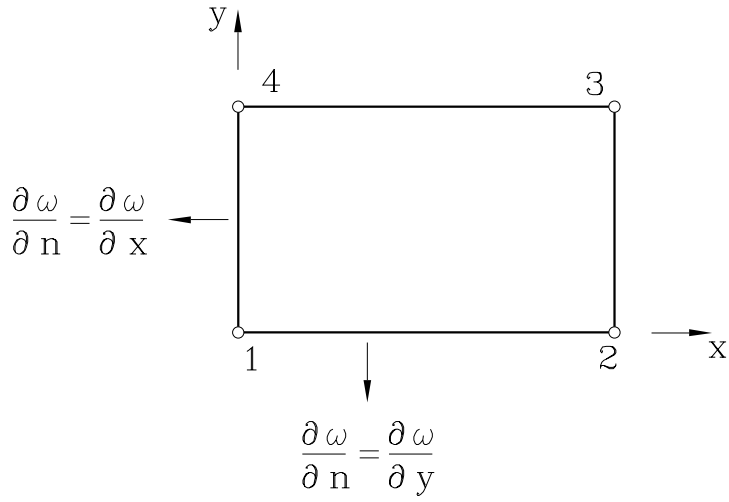

Figure 4 : A four node rectangular plate element.

There are two nodes on side 1-2, hence two values of *w* and it's derivative with respect to *x*. The displacement function, *w*, in equation (33) will have a cubic variation in this direction. However, as the coordinate *y* has a constant value along side 1-2, the slope, θ_y can only have a linear variation. The linear variation, will be obtained from the values of θ_2 at nodes 1 and 2. The polynomial defining the displacement *w*, is a function of *x* only as *y* is constant, therefore no variation of θ_y can be obtained, other than using the values at the nodes. Therefore there is a lack of continuity at the corners. If side 1-4 is examined, it can be deduced, from the above argument that θ_x will have a linear distribution along the element side.

If the [B] matrix is examined, it will be seen that for the strain-displacement relationship, a second order differential of the displacement with respect to *x* and *y* exists. The distribution of θ_y along the side 1-2 is linear dependant on the nodal values of nodes 1 and 2. If this value is differentiated with respect to *x*, then $\frac{\partial^2 w}{\partial x \partial y}$ will be a function of the two nodal values only. Similarly along side 1-4, θ_x is dependant on the two nodal values (at nodes 1 and 4). If this is differentiated with respect to *y* then $\frac{\partial^2 w}{\partial y \partial x}$ will be a function of the nodal values. At node 1, it is impossible to satisfy the condition that $\frac{\partial^2 w}{\partial x \partial y}$ is continuous. Thus, if it is impossible to obtain continuity within the element, then there is no way that continuity across the element boundaries can be achieved, using a simple polynomial. An element which uses such a polynomial is generally known as having ***non-conforming*** shape functions.

2.3.1 Four Node Quadrilateral Element

The shape function for a four noded quadrilateral element with three degrees of freedom at each node, can be obtained by assuming a polynomial trial function for the displacement *w*. This would contain twelve unknowns coefficients,

$$w = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 + \alpha_6 x^3 + \alpha_7 x^2 y + \alpha_8 x y^2 + \alpha_9 y^3 + \alpha_{10} x^3 y + \alpha_{11} x y^3 \tag{34}$$

The coefficients α_0 to α_{11} can be obtained by substituting the boundary conditions at each node into equation (34), thus obtaining twelve simultaneous equations. It is suggested, in some texts that, this method be used to obtain the shape functions. This, however, has the disadvantage that a standard, intrinsic element cannot be defined and the procedure would have to be carried out for every element. The geometry of the element is limited to rectangles to allow numerical integration to be employed. The equations can, and have been solved algebraically, but this is very time consuming and error prone. This does, however allow the use of an intrinsic element, which will aid the numerical implementation of the method.

An alternative method for deriving the Hermitian interpolation for an intrinsic element was presented by El-Zafrany and Cookson. The problem to be solved is given by,

$$\frac{\partial^s}{\partial \xi^s}\frac{\partial^t}{\partial \eta^t}\left(\tilde{w}_{ij}\right) = \frac{\partial^s}{\partial \xi^s}\frac{\partial^t}{\partial \eta^t} w_{ij} \tag{35}$$

where *i* varies from 1 to the number of nodes in ξ direction

j varies from 1 to the number of nodes in η direction

s varies from 0 to the number of derivative required

t varies from 0 to the number of derivative required

$\tilde{w}$ is the approximate function

w is the actual function values at the nodes

It was shown that,

$$\tilde{w} = \sum_{i=1}^{n} \left\{ \sum_{s=0}^{u} H_i^{u,s}(m,\xi) \frac{\partial^s}{\partial \xi^s} \sum_{j=1}^{m} \sum_{t=0}^{v} H_j^{v,t}(n,\eta) \frac{\partial^t}{\partial \eta^t} \right\} w_{ij} \tag{36}$$

where $H_i^{u,s}$ and $H_j^{v,t}$ are one dimensional Hermitian polynomials.

Consider the four node quadrilateral element, shown in *Figure 5*, each node has three degrees of freedom.

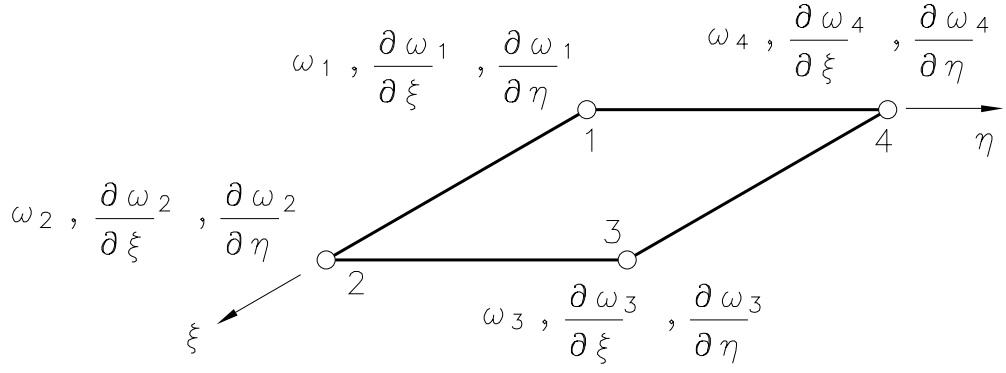

Figure 5 : Four node quadrilateral plate element with degrees of freedom.

If equation (36) is considered, it can be seen that *i* and *j* vary from 1 to 2 (the number of nodes in the ξ and η directions respectively) and *s* and *t* vary from 0 to 1, as the first derivative of *w*, is required. However, this produces a cross derivative term $\frac{\partial^2}{\partial x \partial y}$ when both *s* and *t* have a value of 1. This is not a valid degree of freedom, and hence equation (35) is examined as a series of special cases and superposition is used to obtain the overall results. The case where *t* is zero and *s* varies from 0 to 1 is considered and equation (35) yields,

$$\tilde{w} = w_I\left(w, \frac{\partial w}{\partial \xi}\right) \tag{37}$$

where a function of *w* is obtained over the whole element, a cubic function of $\frac{\partial w}{\partial \xi}$ is obtained over the edges where η is constant and a linear interpolation over the edges where ξ is constant.

The second case considered, where *s* is zero and *t* varies from 0 to 1 gives,

$$\tilde{w} = w_{II}\left(w, \frac{\partial w}{\partial \eta}\right) \tag{38}$$

Thus $\frac{\partial w}{\partial \xi}$ and $\frac{\partial w}{\partial \eta}$ are obtained over the element but *w* has been defined over the element twice. The special case of both *s* and *t* are zero is therefore considered to give,

$$\tilde{w} = w_{III}(w) \tag{39}$$

The interpolation formula for the element can be obtained using the principal of superposition as,

$$\tilde{w} = w_I + w_{II} - w_{III} \tag{40}$$

This is shown graphically in *Figure 6*.

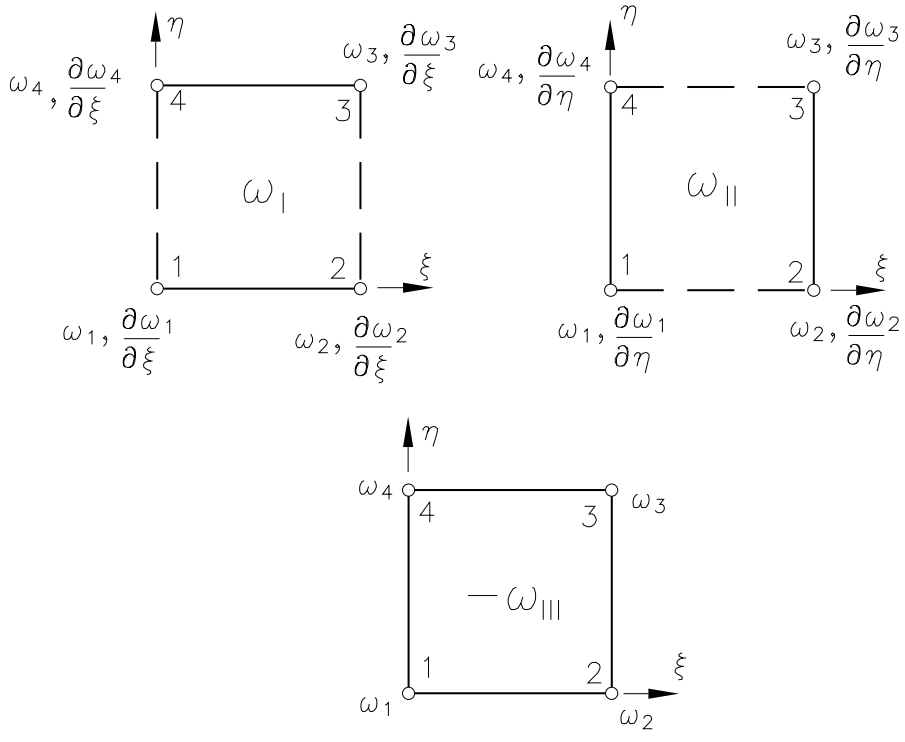

Figure 6 : Graphical representation of shape functions for a thin plate.

Equation (40) can be evaluated to give,

$$\sum_{j=1}^{m} \left[\left\{ H_i^{1,0}(\xi) H_j^{1,0}(\eta) + H_i^{1,0}(\xi) H_j^{1,0}(\eta) - H_i^{1,0}(\xi) H_j^{1,0}(\right. \right. \\ \left. \left. + H_i^{1,1}(\xi) H_j^{1,0}(\eta) \frac{\partial w_{ij}}{\partial \xi} + H_i^{1,0}(\xi) H_j^{1,1}(\eta) \frac{\partial w_{ij}}{\partial \eta} \right] \right. \tag{41}$$

Consider the second term of the above equation,

$$H_i^{1,1}(\xi) H_j^{1,0}(\eta) \frac{\partial w_{ij}}{\partial \xi} \tag{42}$$

The interpolation along the edges where η is constant will use the Hermitian shape functions, ie,

$$\begin{aligned} H_1^{1,1} &\equiv N_2 \\ H_2^{1,1} &\equiv N_4 \end{aligned} \tag{43}$$

where N_2 and N_4 are the one-dimensional shape functions which were defined for the Euler beam element. The interpolation function along the sides of constant, is given in equation (42) by the term,

$$H_j^{1,0}(\eta) \tag{44}$$

which actually varies as a Lagrangian polynomial, determined by the number of nodes along the edge. Thus equation (42) should be written as,

$$H_i^{1,1}(\xi)\,\mathcal{L}_j^n(\eta)\,\frac{\partial w_{ij}}{\partial \xi} \tag{45}$$

where,

$$\mathcal{L}_j^n(\eta) = \prod_{\substack{r=1 \\ r\neq j}}^{n} \left(\frac{\eta - \eta_r}{\eta_j - \eta_r} \right) \tag{46}$$

is a Lagrangian polynomial, defined for shape functions of the one dimensional axial bar element.

A similar procedure is carried out for the other terms of equation (41). Thus equation (41) can be written as

$$\sum_{j=1}^{m} \Big[\Big\{ H_i^{1,0}(\xi)\,H_j^{1,0}(\eta) + H_i^{1,0}(\xi)\,H_j^{1,0}(\eta) - H_i^{1,0}(\xi) H_j^{1,0}(\\ + H_i^{1,1}(\xi) H_j^{1,0}(\eta)\,\frac{\partial w_{ij}}{\partial \xi} + H_i^{1,0}(\xi)\,H_j^{1,1}(\eta)\,\frac{\partial w_{ij}}{\partial \eta} \Big] \tag{47}$$

This produces shape functions for an intrinsic element which allows transformation from the actual element to an element on an intrinsic plane.

The mathematics involved in the complete derivation is beyond the scope of this text, but the derivation given shows the complexity of the problem. There are many modifications made to the basic theory to improve the element performance, these are not covered in this text, but some of the references given discuss these elements in more detail. In the basic theory, it was stated that the slope continuity was not satisfied. The effect of this can be determined using a patch test. It is generally

found that the results obtained using this element are acceptable. It must be pointed out that the basic theory is an approximation and the finite element analysis is a further approximation, therefore the expected level of accuracy obtained from these elements is not high. However, it is found that these elements produce good results for thin plates but as the thickness of the element increase the accuracy decreases. This is because the derivation does not take into account any shear deformation effects.

2.3.2 Three Node Triangular Element

The triangular element can be defined by a series of natural or area coordinates, as shown below in *Figure 7.*

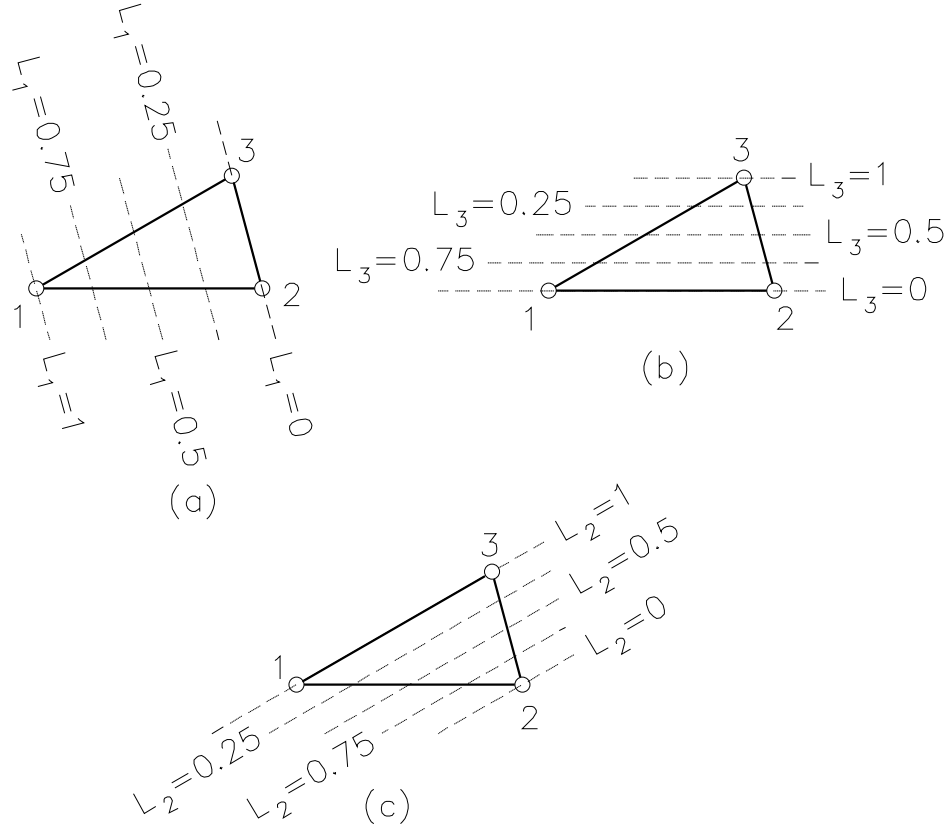

Figure 7 : A three node triangular element defined using area coordinates.

The coordinate of any point within the triangle can be defined in terms of the nodal coordinates and the natural coordinates as,

$$\begin{aligned} x &= L_1 x_1 + L_2 x_2 + L_3 x_3 \\ y &= L_1 y_1 + L_2 y_2 + L_3 y_3 \end{aligned} \tag{48}$$

Natural coordinates are often refereed to as area coordinates and the reason can be seen in *Figure 8*.

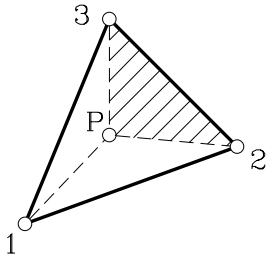

Figure 8 : Triangular element divided into a series of areas.

It can be seen that, for a point *P*,

$$\begin{aligned} L_1 &= \frac{Area\ 23P}{Area\ 123} \\ L_2 &= \frac{Area\ 13P}{Area\ 123} \\ L_3 &= \frac{Area\ 12P}{Area\ 123} \end{aligned} \tag{49}$$

The area of the element is equal to the sum of the areas of the three segments in *Figure 8* which define the area coordinate, thus it can be deduced that,

$$L_1 + L_2 + L_3 = 1 \tag{50}$$

From equations (48), (49) and (50) the area coordinates can be found in terms of the nodal parameters,

$$\begin{aligned} L_1 &= \frac{(x_2 - x_3)(y - y_2) - (y_2 - y_3)(x - x_2)}{2A} \\ L_2 &= \frac{(x_3 - x_1)(y - y_1) - (y_3 - y_1)(x - x_1)}{2A} \\ L_3 &= \frac{(x - x_1)(y_2 - y_1) - (y - y_1)(x_2 - x_1)}{2A} \end{aligned} \tag{51}$$

where *A* is the area of the element.

The area coordinates can be used to define a polynomial for the field function, in this case the displacement *w*. The polynomial requires one term for each degree of freedom of the element. In this case there are three degrees of freedom per node and three nodes per element giving a total of nine terms. The complete cubic polynomial has ten terms, which is one more than required, so generally a mixture of quadric and cubic terms are used to define the polynomial.

$$w = \alpha_0 L_1 + \alpha_1 L_2 + \alpha_2 L_3 + \alpha_3 L_1 L_2 + \alpha_4 L_2 L_3 + \alpha_5 L_3 L_1 + \alpha_6 L_1^2 L_2 + \alpha_7 L_2^2 L_3 + \alpha_8 L_3^2 L_1 \tag{52}$$

The rotation degrees of freedom can be obtained by using the chain rule of partial differentiation.

$$\frac{\partial}{\partial x} = \frac{\partial L_1}{\partial x}\frac{\partial}{\partial L_1} + \frac{\partial L_2}{\partial x}\frac{\partial}{\partial L_2} + \frac{\partial L_3}{\partial x}\frac{\partial}{\partial L_3}$$

$$\frac{\partial}{\partial y} = \frac{\partial L_1}{\partial y}\frac{\partial}{\partial L_1} + \frac{\partial L_2}{\partial y}\frac{\partial}{\partial y} + \frac{\partial L_3}{\partial y}\frac{\partial}{\partial L_3} \tag{53}$$

Equations (53) can easily be evaluated by differentiating equation (52) with respect to L_1, L_2 and L_3 and differentiating equations (51) with respect to *x* and *y*.

The next stage is to substitute the boundary conditions at the nodes into the resulting equations. This gives nine equations for the unknowns α_0 to α_8 and hence the element shape functions can be obtained.

There are various other methods for deriving the shape functions for this element and again this is an area where a great deal of research has been carried out. The process has only been outlined in this text as the mathematics involved are time consuming and error prone and the element has some drawbacks when used for practical applications. Many researchers have tried various modifications to improve the performance of this element. This subject has been covered in various texts in much greater depth than this introductory text.

3 Mindlin Plate Analysis

3.1 Introduction

There are a number of basic criteria which must be satisfied for a three-dimensional structure to be considered as a plate. These criterion are listed in Section 1. The plate theory developed by Mindlin has an additional assumption which is that ***the normal to the mid-surface before deformation remains straight but not necessarily normal after deformation***. This assumption give an approximation for the shear deformation, but does not take into account any warping of the element caused by the shear deformations. In general a correction factor is added in the analysis to take account of warping.

3.2 Mindlin Plate Element

The plate shown in *Figure 9*, has the *x* and *y* axes coincident with the neutral surface, with the *z* axis in the direction of the bottom surface of the plate.

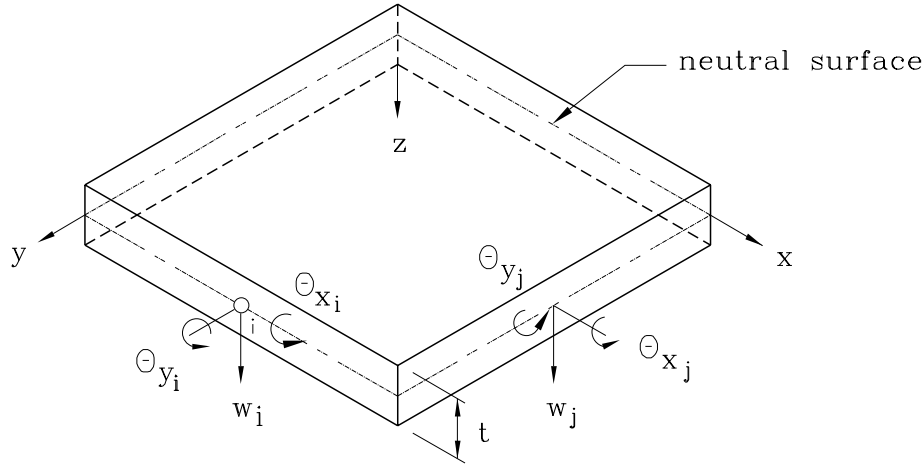

Figure 9 : A plate with axes definition.

At any node such as *i* or *j*, there are three degrees of freedom displacement, two rotations about the in-plane axes and a translation in the global *z* direction.

The theory of total minimum potential energy can be employed to derive the finite element equations. The procedures employed in two and three-dimensional static analysis are also used to derive the

Mindlin plate element. The total potential energy is given by the difference between the work done and the strain energy. The work done on the four node quadrilateral element, shown in *Figure 10* is,

$$= w_1 P_{z_1} + \theta_{x_1} M_{x_1} + \theta_{y_1} M_{y_1} + \ldots\ldots + w_4 P_{z_4} + \theta_{x_4} M_{x_4} + \theta_{y_4} M \tag{54}$$

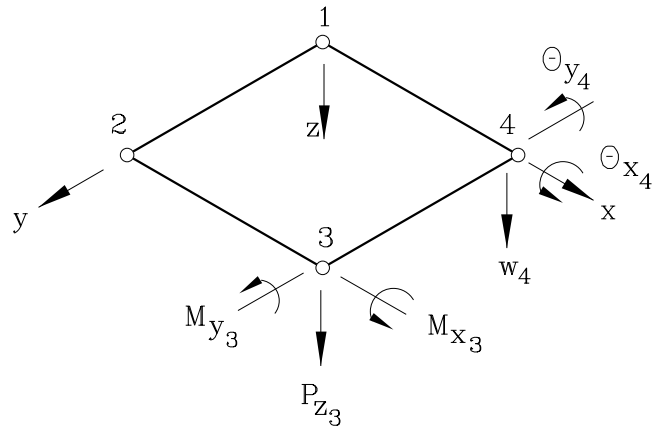

Figure 10 : Nodal work done by a four node quadrilateral element.

This is generally written in matrix form as,

$$W = \{a\}^t \{f\} \tag{55}$$

where,

$$\begin{aligned} \{a\}^t &= \left[w_1 \;\; \theta_{x_1} \;\; \theta_{y_1} \;\; \ldots\ldots \;\; w_4 \;\; \theta_{x_4} \;\; \theta_{y_4} \right] \\ \{f\}^t &= \left[F_{z_1} \;\; M_{x_1} \;\; M_{y_1} \;\; \ldots\ldots \;\; P_{z_4} \;\; M_{x_4} \;\; M_{y_4} \right] \end{aligned} \tag{56}$$

The strain energy for a structure is given by,

$$U = \int_{Vol} \frac{\{\sigma\}\{\epsilon\}}{2} \, dVol \tag{57}$$

where,

$$\begin{aligned} \{\sigma\}^t &= \left[\sigma_x \;\; \sigma_y \;\; \tau_{xy} \;\; \tau_{yz} \;\; \tau_{zx} \right] \\ \{\epsilon\}^t &= \left[\epsilon_x \;\; \epsilon_y \;\; \gamma_{xy} \;\; \gamma_{yz} \;\; \gamma_{zx} \right] \end{aligned} \tag{58}$$

The stresses and strains are related by Hooke's law for linear elastic homogeneous materials. The direct stresses are related to the direct strains by,

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - nu\,\sigma_y\right)$$
$$\epsilon_y = \frac{1}{E}\left(\sigma_y - nu\,\sigma_x\right) \tag{59}$$

These equations can be re-written in matrix form and re-arranged to give,

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\end{Bmatrix} = \frac{E}{1-nu^2}\begin{bmatrix}1 & nu\\ nu & 1\end{bmatrix}\begin{Bmatrix}\epsilon_x\\ \epsilon_y\end{Bmatrix} \tag{60}$$

The shear strains are related to the shear stresses,

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\gamma_{yx} = \frac{\tau_{yz}}{G} \tag{61}$$
$$\gamma_{zx} = \frac{\tau_{zx}}{G}$$

where,

$$G = \frac{E}{2(1-nu)} \tag{62}$$

The strain term γ_{xy}, is due to the bending while the terms γ_{yz} and γ_{zx}, are due to the shear. The bending shear stress is included with the direct bending stress to give,

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-nu^2}\begin{bmatrix}1 & nu & 0\\ nu & 1 & 0\\ 0 & 0 & \frac{1-nu}{2}\end{bmatrix}\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix} \tag{63}$$

This is generally written as,

$$\{\sigma\}_b = [D]_b\{\epsilon\}_b \tag{64}$$

The shear stress-strain relationship is give by,

$$\begin{Bmatrix}\tau_{yz}\\ \tau_{zx}\end{Bmatrix} = \frac{E}{2(1-nu)}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\gamma_{yz}\\ \gamma_{zx}\end{Bmatrix} \tag{65}$$

which is generally written as,

$$\{\sigma\}_s = [D]_s \{\epsilon\}_s \tag{66}$$

The stress-strain relationships are partitioned in such a way to allow the inclusion of a two-dimensional plane stress stiffness matrix into the finite element equation. This gives a more general plate element which allows both in-plane and out-of-plane loading to be applied. The $[D]_b$ matrix for the plane element is identical to the [D] matrix for a plane strain analysis, the partition of the stress-strain relationship means the same matrix is not calculated twice. The inclusion of a plane stress analysis is not considered further in this text.

The strain-displacement (***compatibility***) relationships for small deflections are,

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} \\
\epsilon_y &= \frac{\partial v}{\partial y} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\
\gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}
\end{aligned} \tag{67}$$

At first glance, it may seem that most of the strain terms will be zero as there is no displacements in the *x* and *y* directions, defined as active degrees of freedom for the element. This is true, only for the neutral surface, if *Figure 11* is examined, it can be seen that there is a displacement in the *x* direction at points through the thickness of the plate.

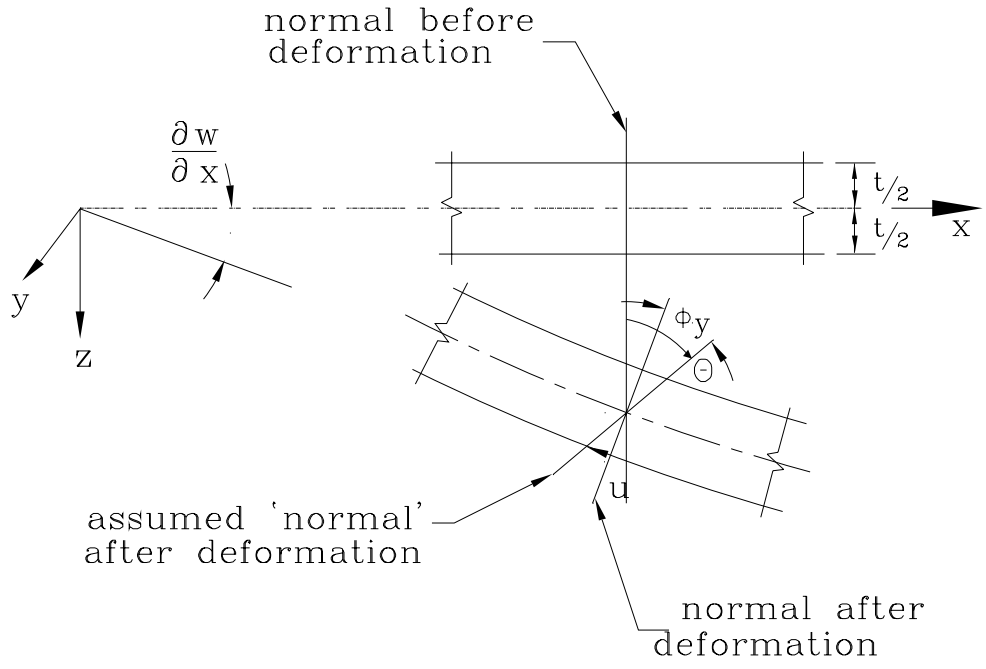

Figure 11 : The component of displacement through the thickness of the plate.

It can be seen from *Figure 11*, that at a distance *z* from the neutral surface, in the displaced section, there is a displacement *u*, due to the application of moment M_y. The moment M_y, causes a negative rotation θ_y and the displacement *u*, is also in the negative direction. It can also be seen that the rotation θ_y has two components, thus,

$$\theta_y = -\left(\frac{\partial w}{\partial x} + \phi_y \right) \tag{68}$$

The displacement *u*, is given by,

$$u = z\theta_y \tag{69}$$

(As θ_y is negative, *u* will automatically be negative) or substituting equation (68), (124) into equation (69),

$$u = -z\left(\frac{\partial w}{\partial x} + \phi_y \right) \tag{70}$$

Similarly *Figure 12*, shows the original and displaced shapes of the plate, with the corresponding rotations and displacements.

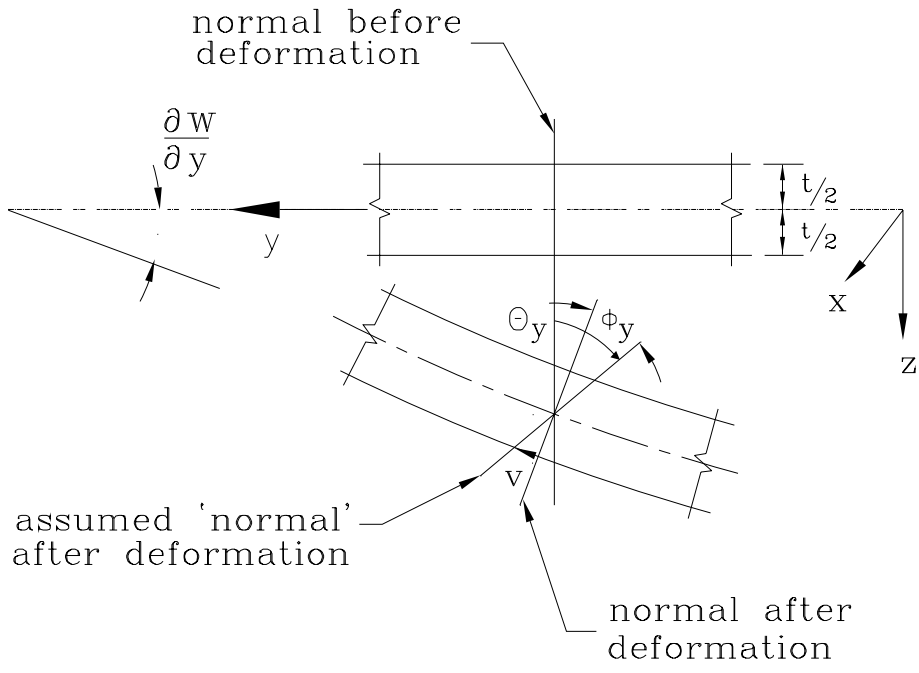

Figure 12 : Displacement components through the thickness of the plate.

It can be seen that a moment M_x causes a negative rotation θ_x, but a positive displacement v at a distance z from the neutral surface. It should also be noted that the slope of the tangent to the displaced neutral surface is negative. The rotation has two components and can be written as,

$$\theta_x = \frac{\partial w}{\partial y} + \phi_y \tag{71}$$

The displacement, v, is therefore,

$$v = -z\,\theta_x \tag{72}$$

(As θ_x is negative, when substituting into equation (72) it will give v as positive) or substituting equations (71) into equation (72) will yield,

$$v = -z\left(\frac{\partial w}{\partial x} + \phi_x\right) \tag{73}$$

It can be seen from equations (70) and (73) that u and v are not wholly dependant on w and can therefore be treated as independent variables. This means that Lagrangian shape functions identical to the ones used for two-dimensional static analysis can be used, in this analysis. The problem is one of $\boldsymbol{C_0}$ continuity, as only the displacements need be continuous between elements.

The displacement at any point on the plate can be specified using nodal parameters and nodal shape

functions,

$$\begin{aligned} w &= \sum_{i=1}^{n} N_i w_i \\ \theta_x &= \sum_{i=1}^{n} N_i (\theta_x)_i \\ \theta_y &= \sum_{i=1}^{n} N_i (\theta_y)_i \end{aligned} \tag{74}$$

The nodal strain components can also be written in terms of the nodal variables and shape functions as,

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(z\theta_y\right) = z\sum_{i=1}^{n}\frac{\partial N_i}{\partial x}\theta_{y_i}$$

$$\epsilon = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(-z\theta_x\right) = -z\sum_{i=1}^{n}\frac{\partial N_i}{\partial y}\theta_{x_i} \tag{75}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(-z\theta_x\right) + \frac{\partial}{\partial y}\left(z\theta_y\right)$$

$$= z\sum_{i=1}^{n}\left\{\frac{\partial N_i}{\partial x}\theta_{y_i} - \frac{\partial N_i}{\partial y}\theta_{x_i}\right\} \tag{76}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\partial}{\partial z}\left(-z\theta_x\right) + \frac{\partial}{\partial y}(w)$$

$$= \sum_{i=1}^{n}\left\{-\theta_x N_i + w_i\frac{\partial N_i}{\partial y}\right\} \tag{77}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial}{\partial z}\left(z\theta_y\right) + \frac{\partial}{\partial x}(w)$$

$$= \sum_{i=1}^{n}\left\{-\theta_y N_i + w_i\frac{\partial N_i}{\partial x}\right\} \tag{78}$$

Equations (75) to (78) can be written in matrix form, for the *ith* node as,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & z\dfrac{\partial N_i}{\partial x} \\ 0 & -z\dfrac{\partial N_i}{\partial y} & 0 \\ 0 & -z\dfrac{\partial N_i}{\partial x} & z\dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial y} & -N_i & 0 \\ \dfrac{\partial N_i}{\partial x} & 0 & N_i \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{x_i} \\ \theta_{y_i} \end{Bmatrix} \tag{79}$$

The displacement vector would actually contain *3n* values for the complete element and the [B] matrix of equation (79) is repeated for every with the subscript of that node. The strain-displacement equation, like the stress-strain equations, are partitioned such that,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & z\dfrac{\partial N_i}{\partial x} \\ 0 & -z\dfrac{\partial N_i}{\partial y} & 0 \\ 0 & -z\dfrac{\partial N_i}{\partial x} & z\dfrac{\partial N_i}{\partial y} \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{x_i} \\ \theta_{y_i} \end{Bmatrix} \tag{80}$$

which is written as,

$$\{\epsilon\}_b = [B_b]\{a\} \tag{81}$$

and,

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_i}{\partial y} & -N_i & 0 \\ \dfrac{\partial N_i}{\partial x} & 0 & N_i \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{x_i} \\ \theta_{y_i} \end{Bmatrix} \tag{82}$$

which is written as,

$$\{\epsilon\}_s = [B_s]\{a\} \tag{83}$$

This partitioning is in line with the partitioning of the stress-strain relationship.

Using equations (64), (66), (81) and (83) the strain energy equation for the element is found to be,

$$\frac{1}{2}\{a\}^t\left[\int_{Vol}[B_b]^t[D_b][B_b]\,dVol\right]\{a\}+\frac{1}{2}\{a\}^t\left[\int_{Vol}[B_s]^t[D_s][B_s]\,dVol\right] \tag{84}$$

The potential energy of the element can therefore be written as,

$$\Pi = \frac{1}{2}\{a\}^t[K]_b\{a\}+\frac{1}{2}\{a\}^t[K]_s\{a\}-\{a\}^t\{f\} \tag{85}$$

where,

$$\begin{aligned}[K]_b &= \int_{Vol}[B_b]^t[D_b][B_b]\,dVol\\ [K]_s &= \int_{Vol}[B_s]^t[D_s][B_s]\,dVol\end{aligned} \tag{86}$$

are the stiffness matrices due to bending and shear respectively.

The finite element equation is obtained by finding the minimum potential energy of the structure, this corresponds to the equilibrium position of the loaded structure and is given by,

$$\frac{\partial\Pi}{\partial\{a\}^t} = 0 \tag{87}$$

Hence, the finite element equation is,

$$\left[K_b+K_s\right]\{a\} = \{f\} \tag{88}$$

The stiffness matrices can be partially evaluated analytically as the integration through the thickness is straight forward. In the case of the bending stiffness matrix, the $[B_b]$ matrix has a constant value of z, thus,

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2[D_b]dz = \frac{h^3}{12}[D_b] \tag{89}$$

Generally the $[D_b]$ matrix is written as,

$$[D_b] = \frac{Eh^3}{12(1-nu)}\begin{bmatrix}1 & nu & 0\\ nu & 1 & 0\\ 0 & 0 & \frac{1-nu}{2}\end{bmatrix} \tag{90}$$

In a similar way, the $[B_s]$ matrix for the shear components is not a function of z, hence,

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} [D_s]\,dz = h\,[D_s] \tag{91}$$

Again the matrix is modified to,

$$[D_s] = \frac{\alpha E h}{2(1+nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{92}$$

where α is a correction factor introduced to take account of the non-uniform shear strain distribution and warping. For isotropic materials, $\alpha = 5/6$.

The element can be transformed to an intrinsic coordinate system in a similar manner to the two-dimensional plane stress analysis. This allows numerical integration to be carried out to evaluate the stiffness matrix.

The Mindlin plate bending element has advantages over the other plate bending elements, these are mainly the ease of derivation and it's extension to a five degree of freedom element including in-plane loading.

The Mindlin plate bending element has a disadvantage, in the fact that it becomes less accurate than the Kirchhoff element as the plate thickness decreases. This is due to ***shear locking***. Generally the problem can be overcome using ***reduced*** numerical integration or ***selective*** numerical integration on the offending shear term.

4 Shell Elements

4.1 Introduction

There are many examples of shell structures in engineering, including the outer surfaces of most modes of powered transport. The bodies of cars, aeroplanes, trains and space vehicles are all shells. The shell offers a number of advantages over other engineering forms, such as high strength to weight ratio's, stiffness can be *built-in* in the geometry of the shell or by adding shell stiffeners, shells can be pressed into aerodynamic shapes. Shells can be distinguished from plates by the geometry, shells are either singly or doubly curved. The shell thickness is small compared with the other dimensions.

The analysis of shell structures is of vital importance due to their use in safety critical situation, especially the in the aerospace industry. The geometry of the problem makes the analysis of shells particularly complex. There are three main approaches to the finite element analysis of shells and countless elements have been developed over the years. Many different displacement fields have been assumed and different strain-displacements relationships assumed. There is often little agreement on which of the terms of the strain-displacement are negligible. This is the reason that so many different shell element have been developed.

Shell elements fall into one of three categories,

1. Facet (Flat) shell elements
2. Solid shell elements
3. Curved shell elements

The facet shell element is a flat element which approximates the curved surface by a series of facets. Only triangular elements can give a true facet representation of a curved surface. Indeed quadrilateral elements can suffer from problems when used to approximate a shell structure if the elements are co-planar. The facet shell elements are generally flat plat elements combined with plane stress elements. The most obvious problem with using a facet representation of a curved surface is that discretisation errors automatically occur. There are other difficulties which arise in the formulation of these elements. The slope terms in the assumed displacement field are discontinuous within the element. The continuity of slopes between elements is also lost and therefore the ***compatibility*** condition is not satisfied. The results obtained from these elements tend to be on the upper bound of error limits or in many cases are inaccurate. The advantage of this element is that its formulation is perhaps the

easiest. In the following section the derivation of a thin and thick facet shell element is presented. These are based on the Kirchhoff and Mindlin plate bending elements.

The solid shell elements are based on second order three-dimensional solid elements. Using second order elements allows the curved geometry to be modelled more accurately. As the shell thickness is small compared with the other dimensions the problem of aspect ratio error occurring immediately springs to mind when using a three-dimensional element. This is overcome by assuming that no direct strain occurs through the element thickness. This allows the mid-side nodes through the thickness to be removed. Therefore a twenty node brick element becomes a sixteen element. This is often referred to as a ***degenerate*** solid element. Transverse shear deformation is included within this element as the displacement is calculated on the top and bottom surface of the shell, and are not tied to the slope of the mid-surface of the shell. Even with the removal of the mid-side nodes through the surface of the element, aspect ratio errors can still pose a problem and this type of element is mainly used for thick shell analysis. Another drawback of this element is the large number of degrees of freedom and hence large computer memory requirements. The results for these elements are acceptable, although not as accurate as the curved shell element. The accuracy is improved using reduced integration schemes. The attraction of these elements are that they can be formulated without having to use classical shell theory. The three-dimensional element is included in most finite element packages and it is relatively straight forward to modify the code to include a solid shell element. These will not be dealt with in the body of this introductory text.

The curved shell elements, although difficult to formulate, have been developed to overcome the inaccuracy of the other shell elements. As there is no accepted standard strain-displacement relationship (after neglecting some of the terms) these elements tend to be derived for particular geometries. The geometric transformation from a two-dimensional curved surface in three-dimensional space to a standard element also limits the type of geometry that the elements can be applied to. Thus in many cases these elements do not have general applicability. These elements approximate the geometry quite closely if not exactly.

5 Facet Shell Elements

5.1 Introduction

The thin shell element is a combination of the two-dimensional plane stress element and the Kirchhoff plate bending element and as such can carry both in-plane and out-of-plane loading. The limitations of the Kirchhoff plate element, regarding the accuracy of the results and the plate thickness also applies to the shell element. The shell element is a two-dimensional element in three-dimensional space. Each element will have a local system of axes, but the model will be in a global coordinate system. The finite element equations are obtained for the element in terms of the local axes and a transformation from this axis system to the global axis system must be carried out. Similarly there is a thick shell element based on the Mindlin plate element and a plane stress element.

The plane stress element has two degrees of freedom, displacements *u* and *v* in the global *x* and *y* directions. The plate bending element has three degrees of freedom, one displacement *w*, and two rotations θ_x and θ_y about the global *x* and *y* axes respectively. Both of the rotations are functions of the displacements *w*. Therefore the combined element has five of the six possible degrees of freedom. The sixth degree of freedom θ_z, is included in the equations, although it has zero stiffness and is not a required to calculate the strain components, to simplify the transformation of the coordinate system.

The element consists of a number of nodes which define the element geometry. These nodes are placed on the mid-surface of the element and this is where the loads are applied.

5.2 Kirchhoff Shell Element

The in-plane displacements have two components. One resulting from the in-plane loading, this is given by,

$$u_s = \sum_{i=1}^{n} N_i u_i$$
$$v_s = \sum_{i=1}^{n} N_i v_i \qquad (93)$$

and a component due to the bending moments,

$$u_b = -z\frac{\partial w}{\partial z} = z\theta_y$$
$$v_b = -z\frac{\partial w}{\partial y} = -z\theta_x \tag{94}$$

The other degrees of freedom can be expressed as,

$$w = \sum_{i=1}^{n}\left(a_i w_i + b_i\frac{\partial w_i}{\partial x} + c_i\frac{\partial w_i}{\partial y}\right) \tag{95}$$

The strain matrix is partitioned into two, the first partition contains the membrane components and the second the bending components.

$$\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix}\dfrac{\partial u_s}{\partial x}\\ \dfrac{\partial v_s}{\partial y}\\ \dfrac{\partial u_s}{\partial y}+\dfrac{\partial v_s}{\partial x}\end{Bmatrix} - z\begin{Bmatrix}\dfrac{\partial^2 w}{\partial x^2}\\ \dfrac{\partial^2 w}{\partial y^2}\\ 2\dfrac{\partial^2 w}{\partial x\partial y}\end{Bmatrix} \tag{96}$$

which is commonly expressed as,

$$\{\epsilon\} = \{\epsilon_s\} + \{\epsilon_b\} \tag{97}$$

The strains can now be written in terms of the displacements, for the membrane partition at the *ith* node,

$$\{\epsilon_b\} = \begin{bmatrix}\dfrac{\partial N_i}{\partial x} & 0 & 0 & 0 & 0 & 0\\ 0 & \dfrac{\partial N_i}{\partial y} & 0 & 0 & 0 & 0\\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_i}{\partial x} & 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ w_i\\ \theta_{x_i}\\ \theta_{y_i}\\ \theta_{z_1}\end{Bmatrix} \tag{98}$$

and for the bending partition at the *ith* node,

$$\{\epsilon_b\} = \begin{bmatrix} 0 & 0 & \frac{\partial^2 a_i}{\partial x^2} & \frac{\partial^2 c_i}{\partial x^2} & -\frac{\partial^2 b_i}{\partial x^2} & 0 \\ 0 & 0 & \frac{\partial^2 a_i}{\partial y^2} & \frac{\partial^2 c_i}{\partial y^2} & -\frac{\partial^2 b_i}{\partial y^2} & 0 \\ 0 & 0 & 2\frac{\partial^2 a_i}{\partial x \partial y} & 2\frac{\partial^2 c_i}{\partial x \partial y} & -2\frac{\partial^2 b_i}{\partial x \partial y} & 0 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \theta_{x_i} \\ \theta_{y_i} \\ \theta_{z_i} \end{Bmatrix} \tag{99}$$

The stresses are also partitioned,

$$\{\sigma\} = \{\sigma_s\} + \{\sigma_b\} \tag{100}$$

From the derivation of the plane stress and plate bending elements, it can be seen that the following stress-strain relationship exists,

$$\begin{aligned} \{\sigma_s\} &= [D]\{\epsilon_s\} \\ \{\sigma_b\} &= [D]\{\epsilon_b\} \end{aligned} \tag{101}$$

where,

$$[D] = \frac{E}{1-nu^2}\begin{bmatrix} 1 & nu & 0 \\ nu & 1 & 0 \\ 0 & 0 & \frac{1-nu}{2} \end{bmatrix} \tag{102}$$

The [D] matrix is identical for both the plane stress and the Kirchhoff plate bending element. The strain energy equation can now be written as,

$$\begin{aligned} \Lambda_s &= \frac{1}{2}\{u\}^t \left[\int_{Vol} [B_s]^t [D][B]\, dVol\right]\{u\} \\ \Lambda_b &= \frac{1}{2}\{u\}^t \left[\int_{Vol} [B_s]^t [D][B]\, dVol\right]\{u\} \end{aligned} \tag{103}$$

Upon minimisation of the total potential energy equation, the finite element equation is obtained,

$$[K]\{a\} = \{f\} \tag{104}$$

where,

$$
\begin{aligned}
[K] &= [K_s] + [K_b] \\
[K_s] &= \int_x \int_x [B_s]^t [D] [B_s] \, t \, dy \, dx \\
[K_b] &= \int_x \int_y [B_b]^t [D] [B_b] \frac{t^3}{12} \, dy \, dx
\end{aligned}
\tag{105}
$$

The above equations are for a single element in terms of the local coordinate system. This must be transformed to the global coordinate system, thus the finite element equation becomes,

$$[T]^t [K] [T] \{a\} = \{f\} \tag{106}$$

where,

$$[T] = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \tag{107}$$

where,

$$[C] = \begin{bmatrix} C_{x_l x} & C_{x_l y} & C_{x_l z} \\ C_{y_l x} & C_{y_l y} & C_{y_l z} \\ C_{z_l x} & C_{z_l y} & C_{z_l z} \end{bmatrix} \tag{108}$$

This element has a drawback in that under certain circumstances the stiffness matrix becomes singular. This is due to the fact that the degree of freedom θ_z has no stiffness associated with it. This problem may be overcome by restraining all the θ_z components or by giving a stiffness to θ_z. The second option is acceptable as it results in,

$$k \theta_z = 0 \tag{109}$$

which results in θ_z being zero. The value of k must be comparable with the other components of stiffness to avoid ill-conditioning. The degree of freedom θ_z, has an associated load M_z which can not actually be applied.

5.3 Mindlin Shell Element

The Mindlin shell element is a combination of a two-dimensional plane stress element and the Mindlin plate element. The plane stress element is used to cater for in-plane loading and has degrees of freedom *u* and *v*. The Mindlin plate element allows out-of-plane loading and bending loads to be

applied. The plate element has degrees of freedom w, θ_x, θ_y. The finite element equation for plane stress is,

$$[K_s]\{a_s\} = \{f_s\} \tag{110}$$

where,

$$\begin{aligned} \{a_s\}^t &= \begin{bmatrix} u & v \end{bmatrix} \\ \{f_s\}^t &= \begin{bmatrix} P_x & P_y \end{bmatrix} \end{aligned} \tag{111}$$

The finite element equation for the Mindlin plate element is,

$$[K_b]\{a_b\} = \{f_b\} \tag{112}$$

where,

$$\begin{aligned} \{a_b\}^t &= \begin{bmatrix} w & \theta_x & \theta_y \end{bmatrix} \\ \{f_b\}^t &= \begin{bmatrix} P_z & M_x & M_y \end{bmatrix} \end{aligned} \tag{113}$$

The element stiffness matrix for plane stress analysis is derived in Sections 1.1 of the ***Static I : Two Dimensional Analysis*** booklet and the element stiffness matrix for the plate element is derived in Section 3.2 of this booklet.

The finite element equation for the Mindlin thick shell element is obtained by combining the plane stress and plate bending equations, thus,

$$\begin{bmatrix} K_s & 0 \\ 0 & K_b \end{bmatrix} \begin{Bmatrix} a_s \\ a_b \end{Bmatrix} = \begin{Bmatrix} f_s \\ f_b \end{Bmatrix} \tag{114}$$

The vector of displacements contains all possible degrees of freedom at a node except θ_z. This degree of freedom is not required for the strain and hence the stress calculations. There is therefore no stiffness associated with this degree of freedom. It is however useful to add this degree of freedom to the element equation. Thus,

$$0\,\theta_z = 0 \tag{115}$$

This is combined with equation (114) to give,

$$\begin{bmatrix} K_s & 0 & 0 \\ 0 & K_b & 0 \\ 0 & 0 & 0 \end{bmatrix} \{a\} = \{f\} \tag{116}$$

where,

$$\begin{aligned} \{a\}^t &= \begin{bmatrix} u & v & w & \theta_x & \theta_y & \theta_z \end{bmatrix} \\ \{f\}^t &= \begin{bmatrix} P_x & P_y & P_z & M_x & M_y & M_z \end{bmatrix} \end{aligned} \tag{117}$$

It should be noted that the load, M_z, is a fictitious load and as such cannot actually be applied to the structure. The reason for adding the extra degree of freedom to the finite element equation, is that a transformation from the local element coordinate system to the global coordinate system is required. The shell element is a two-dimensional element, but is defined in three-dimensional space (compare with the beam space element). This is shown below in *Figure 13*.

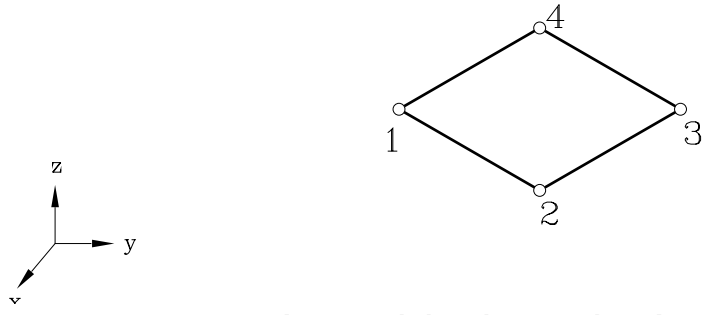

Figure 13 : Two-dimensional plate element in three-dimensional space.

The element must be projected onto the global *x* - *y* plane, so it can then be mapped to an intrinsic element, to allow the element stiffness matrix to be calculated. This transformation is carried out using a transformation matrix, which consists of the direction cosines between the local and global axes. The method presented can be used for any shape element in general three-dimensional space.

The element is defined in terms of the nodes, which are given their position using global coordinates. The first stage is to define the local axis system. The first local axis defined is generally the x_l axis and is defined using nodes 1 and 2. This can be seen in *Figure 14*,

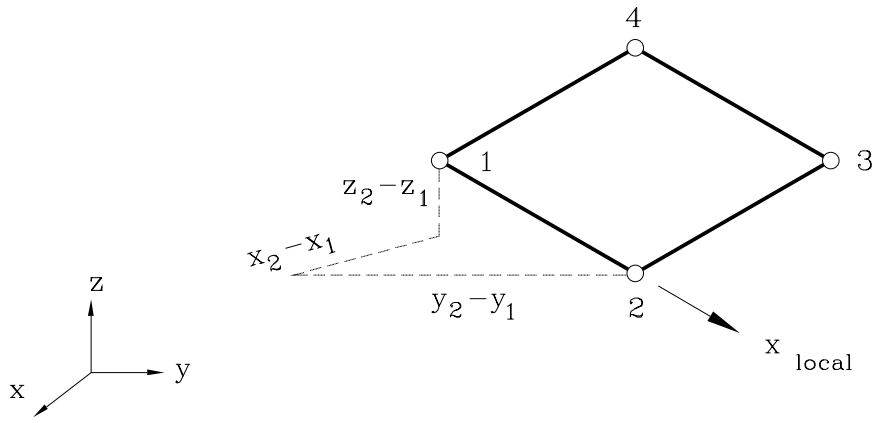

Figure 14 : Definition of local axes of a plate element in three-dimensional space.

The direction cosines are given by the ratio of the component lengths and the actual length between nodes 1 and 2. Therefore,

$$C_{x_l x} = \frac{x_2 - x_1}{L_{1\text{-}2}}$$

$$C_{x_l y} = \frac{y_2 - y_1}{L_{1\text{-}2}} \qquad (118)$$

$$C_{x_l z} = \frac{z_2 - z_1}{L_{1\text{-}2}}$$

where,

$$L_{1\text{-}2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad (119)$$

The next stage is to obtain the local *z* axis direction cosines. This is done by calculating the direction cosines of the side 1-4. (This would be 1-3 if a triangular element were being considered). This is done in exactly the same manner as the direction cosines of sides 1-2. The vector of the side 1-4 is not necessarily perpendicular to side 1-2, as the element may be skew. A triangular element will almost definitely not have perpendicular sides. Thus, this vector cannot be taken as the local *y*-axis. The vectors of sides 1-2 and 1-4 do, however, lie on the same plane, the plane of the element. If the vector cross product is obtained then this will give the direction cosines for the *z* axis.

$$\mathrm{v}_{1\text{-}2} \times \mathrm{v}_{1\text{-}4} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} \qquad (120)$$

The direction cosines are found to be,

$$C_{z_l x} = \frac{(y_2 - y_1)(z_4 - z_1) - (z_2 - z_1)(y_4 - y_1)}{A}$$

$$C_{z_l y} = \frac{(x_2 - x_1)(z_4 - z_1) - (z_2 - z_1)(x_4 - x_1)}{A} \qquad (121)$$

$$C_{z_l z} = \frac{(x_2 - x_1)(y_4 - y_1) - (x_4 - x_1)(y_2 - y_1)}{A}$$

where A is the area of the parallelogram defined by the two vectors. The area of the parallelogram is equivalent to the length of the vector, perpendicular to the area.

The direction cosines of the local x and z axes are now known and only the y axis need to be defined. The cross product of these two vectors will now define the local y axis. The equations previously employed are used to obtain the direction cosines of the y axis. This gives the transformation matrix,

$$[C] = \begin{bmatrix} C_{x_l x} & C_{x_l y} & C_{x_l z} \\ C_{y_x x} & C_{y_l y} & C_{y_l z} \\ C_{z_l x} & C_{z_l y} & C_{z_l z} \end{bmatrix} \qquad (122)$$

The displacements and loads can now be transformed from the local coordinate system to the global coordinate system,

$$\{a\}_g = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \{a\}_l \qquad (123)$$

and,

$$\{f\}_g = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \{f\}_l \qquad (124)$$

Thus, equations (123) and (68), (124) can be substituted into equation (114) to give the finite element equation in terms of the global coordinate system,

$$[T]^t [K] [T] \{a\}_g = \{f\}_g \qquad (125)$$

where,

$$[T] = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \qquad (126)$$

and,

$$[K] = \begin{bmatrix} k_s & 0 & 0 \\ 0 & k_b & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{127}$$

This is the finite element equation for the Mindlin shell element.

There is, however a drawback to this derivation, which in certain cases makes the stiffness matrix singular. This is due to the inclusion of zero stiffness for the θ_z degree of freedom. In some cases the singularity is easily spotted in the stiffness matrix by the finite element program, however if the transformation matrices are used to transform the local to global coordinate system, the singularity may be hidden.

There are two suggested methods for dealing with this singularity. The first is to restrain all θ_z degrees of freedom and remove them from the assembled global stiffness matrix in the reduction process before the equation is solved. This would be carried out by the program and there would be no need for the user to restrain these degrees of freedom.

The second method of dealing with this problem is to give the degree of freedom a stiffness. This would be done by the program and would be transparent to the user. This would give,

$$k\theta_z = 0 \quad \Rightarrow \quad \theta_z = 0 \tag{128}$$

As θ_z is not associated with any of the strain components (and hence the stresses) this does not affect the results and is therefore perfectly acceptable. The value of the stiffness assigned to this degree of freedom should be of comparable magnitude to the other stiffness values in the matrix, to avoid problem with ill conditioning.

6 Curved Shell Elements

6.1 Ahmad Element

The Ahmad element is a curved shell element which approximates the geometry of the structure more accurately than a facet shell element. Curved shell elements by their very nature are higher order elements and unlike faceted elements allow for continuity in the geometry between elements. The Ahmad shell element, like the Mindlin plate element assumes that the normal to the slope after deformations remains straight, but not necessarily normal. This allows the effects of shear deformation to be accounted for and thus this element is used mainly for the analysis of thick shells. It can, however, be used for the analysis of thin shell structures providing that a selective quadrature scheme is employed to carry out numerical integration. In the derivation of the element it is assumed that the strain energy corresponding to the stresses perpendicular to the mid-surface is ignored. The Ahmad element is defined in such a way that, if the shape functions are compatible then the displacement compatibility between the elements is maintained.

The Ahmad element is included in this text because it is precursor to the Semi-Loof shell element, which is the most general shell element available to date. The Semi-Loof element is discussed in more depth in Section 6.2.

A curved shell element is shown in *Figure 15*. The nodes describing the element lie on the mid-surface of the shell and are defined in terms of the global cartesian coordinates.

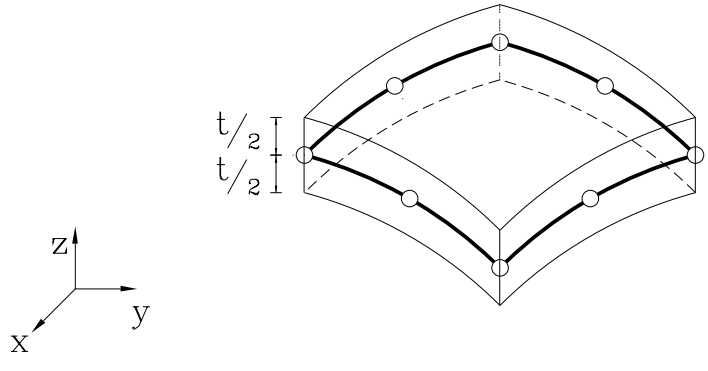

Figure 15 : A doubly curved Ahmad shell element.

It can be seen from *Figure 15* that the element orientation is such that none of the element sides are aligned with any of the global axes. In the derivation of facet shell elements a set of local cartesian

coordinates are defined and a transformation relationship used to convert components of displacement and load from the local to global coordinate systems. This is a relatively straight forward process as the facet element is a flat element. The Ahmad element is doubly curved and local cartesian axes defined at a particular point in the element will have little relevance to any other point in the element. The mid-surface of the element can be defined using curvilinear intrinsic coordinates ξ and η. The linear intrinsic coordinate ζ is used in the direction of the shell thickness. The intrinsic coordinates have values between -1 and 1 (this can also be between 0 and 1 if a modified approach is adopted. In this derivation it the modified approach is used. Therefore the mid-surface of the element lies on a plane where ζ has a constant value of 0.5. The intrinsic coordinate system is shown in *Figure 16*.

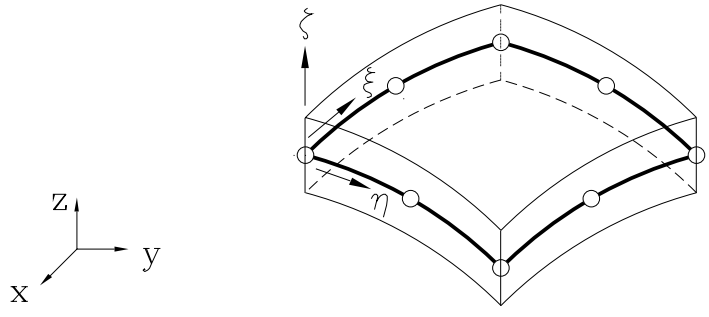

Figure 16 : Intrinsic coordinate system for curved shell element.

Using the curvilinear coordinates a local cartesian coordinate system can be defined for any point on the shell surface. A unit vector parallel to the intrinsic ζ axis at a point in the element is taken as the local cartesian z_l axis. These normals are both perpendicular to the plane of the element.

The finite element equation for the Ahmad shell element can be derived using the theory of minimum total potential energy. The potential energy is the difference between the strain energy stored in the deformed structure and the work done by the external loads.

$$\Pi = U - W \tag{129}$$

The loads and moments are applied to the element at the nodes, therefore the work done by these loads is the sum of the loads and moments multiplied by the nodal displacements and rotations. The nodal displacements of the element are defined in terms of the global cartesian axes directions. The displacement components are the translations *u*, *v* and *w* in the directions of the global *x*, *y*, and *z* axes respectively. A transformation matrix can be defined relating each set of local axes to the global axes. The rotations can only be defined in terms of the local cartesian axes. The rotation about the local x_l axis is taken as β and about the local y_l axis the rotation is α. The degrees of freedom at a node are

shown in *Figure 17*. The forces and moments are defined in a similar manner to the displacements.

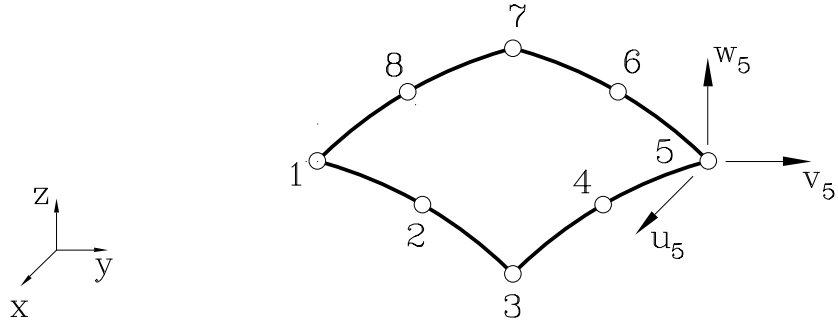

Figure 17 : Displacement components at a node.

Thus the work term can be written as,

$$W = \{a\}^t \{f\} \tag{130}$$

where,

$$\{f\} = \begin{Bmatrix} P_{x_1} \\ P_{y_1} \\ P_{z_1} \\ M_{x_1} \\ M_{y_1} \\ . \\ M_{y_n} \end{Bmatrix} \qquad \{a\} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \alpha_1 \\ \beta_1 \\ . \\ \beta_n \end{Bmatrix} \tag{131}$$

The strain energy for a linear elastic structure subjected to small displacements is given by,

$$U = \int_{Vol} \frac{\{\sigma\}^t \{\epsilon\}}{2} \, dVol \tag{132}$$

where for a shell element,

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \qquad \{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{133}$$

The strain energy of the element is written in terms of stresses and strains in the global coordinate system, they could have as easily been written in terms of the local coordinate system. The

transformation relationship between the local and global coordinate systems used for the displacements can be used for the stresses and strains. It is for the user to decide whether the stress results are more useful in the local or global coordinate system.

For an isotropic, homogeneous linear elastic material the stresses are related to the strains by Hooke's law,

$$\{\sigma\} = [D]\{\epsilon\} \tag{134}$$

where,

$$[D] = \frac{E}{\left(1-nu^2\right)}\begin{bmatrix} 1 & nu & 0 & 0 & 0 \\ nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-nu}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-nu}{2} \end{bmatrix} \tag{135}$$

The elasticity matrix [D] is independent of the coordinates (this is not the case if the material is not isotropic) and can be used to relate the local or global stress to the local or global strains. The strain energy equation (132) can now be written as,

$$U = \frac{1}{2}\int_{Vol} \{\epsilon\}^t [D]\{\epsilon\}\, dVol \tag{136}$$

Previously in this derivation there have been a number of references to the relationship between the local and global coordinate systems without any physical definitions. It would be useful to provide these relationships before going on to examine the strain-displacement relationships.

It has been shown in various derivations that any point within an element can be expressed in terms of the nodal coordinates and shape functions. Therefore any point on the mid-surface of the plane can be expressed as,

$$\begin{aligned} x_m &= \sum_{i=1}^{n} N_i x_i \\ y_m &= \sum_{i=1}^{n} N_i y_i \\ z_m &= \sum_{i=1}^{n} N_i z_i \end{aligned} \tag{137}$$

where N_i are two-dimensional Lagrangian shape functions.

In equation (137) the global coordinates of the element are expressed in terms of the intrinsic coordinates, therefore the intrinsic axes can be expressed as vectors in the global cartesian space,

$$\begin{aligned}\vec{\xi} &= \frac{\partial x}{\partial \xi}\underline{i} + \frac{\partial y}{\partial \xi}\underline{j} + \frac{\partial z}{\partial \xi}\underline{k} \\ \vec{\eta} &= \frac{\partial x}{\partial \eta}\underline{i} + \frac{\partial y}{\partial \eta}\underline{j} + \frac{\partial z}{\partial \eta}\underline{k} \\ \vec{\zeta} &= \vec{\xi} \,.\, \vec{\eta}\end{aligned} \tag{138}$$

The vector defining the ζ axis is obtained by taking the vector cross product of the other two vectors, as it is perpendicular to the plane of these two vectors. The direction cosines of the vector can be obtained by dividing the components of the vector by the overall length of the vector. Thus, the cross product of the two vectors is,

$$\vec{\zeta} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{vmatrix} \tag{139}$$

Evaluating equation (139) results in,

$$\left(\frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \xi}\right)\underline{i} + \left(\frac{\partial z}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial z}{\partial \eta}\frac{\partial x}{\partial \xi}\right)\underline{j} + \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\right) \tag{140}$$

Defining,

$$\begin{aligned} a &= \frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \xi} \\ b &= \frac{\partial z}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial z}{\partial \eta}\frac{\partial x}{\partial \xi} \\ c &= \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \end{aligned} \tag{141}$$

then equation (140) can be re written as,

$$\vec{\zeta} = a\,\underline{i} + b\,\underline{j} + c\,\underline{k} \tag{142}$$

The length of the vector, $\vec{\zeta}$, is given by,

$$L_\zeta = \sqrt{a^2 + b^2 + c^2} \tag{143}$$

The direction cosines of the vector are therefore,

$$\begin{aligned} l_3 &= \frac{a}{L_\zeta} \\ m_3 &= \frac{b}{L_\zeta} \\ n_3 &= \frac{c}{L_\zeta} \end{aligned} \tag{144}$$

The direction cosines for the z_l axis are denoted by the subscript 3, as the direction cosines of the x_l and y_l axes are denoted by subscripts 1 and 2 respectively. A unit vector in the direction of the $\vec{\zeta}$ can be expressed as,

$$\underline{\zeta} = l_3 \underline{i} + m_3 \underline{j} + n_3 \underline{k} \tag{145}$$

The local z_l axis at any point in the element is taken as the normal to the mid-surface. The unit vector given in equation (145) is normal to the mid-plane so therefore describes the local z_l axis. It can be seen from *Figure 15* and *Figure 16*, that at the lower surface of the shell element, where ζ has a value of zero, that the coordinate z_l has a value equal to negative half of the thickness. Similarly, on the top surface of the shell, where ζ has a value of unity, that the coordinate z_l has a value equal to half of the thickness. It can therefore, be deduced that,

$$z_l = (\zeta - 0.5)\, t \tag{146}$$

This can also be written using equation (145) as,

$$\underline{z}_l = l_3 \underline{i} + m_3 \underline{j} + n_3 \underline{k} = (\zeta - 0.5)\, t \tag{147}$$

Equation (147) defines the shell thickness as components in the global coordinate system. Therefore, the global coordinates of any point within the shell can be written as,

$$x = \sum_{i=1}^{n} N_i x_i + \left(\zeta - \frac{1}{2} \right) t\, l_3$$

$$y = \sum_{i=1}^{n} N_i y_i + \left(\zeta - \frac{1}{2} \right) t\, m_3 \qquad (148)$$

$$z = \sum_{i=1}^{n} N_i z_i + \left(\zeta - \frac{1}{2} \right) t\, n_3$$

where l_3, m_3, and n_3 are the direction cosines of the unit vector defining the z_l axis.

The other local axes can now be defined. It is possible to define an infinite number of unit vectors which are perpendicular to the local z_l axis, therefore some decision must be made to define the other axes. In this case it is simplest to use the global cartesian axes as a basis for defining the local x_l and y_l axes. If the z_l axis is parallel to the global cartesian x axis then the local x_l axis is taken as being perpendicular to both the global y axis and the local z_l axis. The direction cosines of the local x_l axis can be obtained by taking the cross product of the two vectors and dividing by the length of the resulting vector, therefore,

$$\underline{x}_l = \frac{\underline{j} \cdot \underline{z}_l}{|\underline{j} \cdot \underline{z}_l|} \qquad (149)$$

The cross product of equation (149) is,

$$\underline{j} \cdot \underline{z}_l = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 0 \\ l_3 & m_3 & n_3 \end{vmatrix} \qquad (150)$$

Thus the vector of the local x_l axis can be written as,

$$\underline{x}_l = \frac{n_3}{\sqrt{m_3^2 + n_3^2}} \underline{i} + 0\, \underline{j} + \frac{m_3}{\sqrt{m_3^2 + n_3^2}} \underline{k} \qquad (151)$$

The direction cosines of this vector are therefore,

$$l_1 = \frac{n_3}{\sqrt{m_3^2 + n_3^2}}$$

$$m_1 = 0 \tag{152}$$

$$n_1 = \frac{m_3}{\sqrt{m_3^2 + n_3^2}}$$

The unit vector in the direction of the local y_l axis can be obtained as it is perpendicular to both the x_l and z_l axes.

$$\underline{y}_l = \frac{\underline{x}_l \cdot \underline{z}_l}{|\underline{x}_l \cdot \underline{z}_l|} \tag{153}$$

In most cases the local z_l axis will not be parallel to the global x axis. In such cases the local x_l axis, will be assumed to be perpendicular to the global x axis and the local z_l axis. In this case the unit vector defining the axis is,

$$\underline{x}_l = \frac{\underline{i} \cdot \underline{z}_l}{|\underline{i} \cdot \underline{z}_l|} \tag{154}$$

The cross product of the vectors in equation (154) is,

$$\underline{i} \cdot \underline{z}_l = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 0 \\ l_3 & m_3 & n_3 \end{vmatrix} \tag{155}$$

Thus the vector of the local x_l axis, in this case is,

$$\underline{x}_l = 0\,\underline{i} - \frac{n_3}{\sqrt{m_3^2 + n_3^2}}\underline{j} + \frac{m_3}{\sqrt{m_3^2 + n_3^2}}\underline{k} \tag{156}$$

The direction cosines of this vector are therefore,

$$l_1 = 0$$

$$m_1 = -\frac{n_3}{\sqrt{m_3^2 + n_3^2}} \qquad (157)$$

$$n_1 = \frac{m_3}{\sqrt{m_3^2 + n_3^2}}$$

The unit vector defining the local y_l axis is again found using equation (153).

The direction cosines represent the angles between the global and local axes and can be used to form a transformation matrix between the two coordinate systems. The transformation matrix is given by,

$$[T] = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \qquad (158)$$

It was previously stated that at each node there are three displacement components in the directions of global cartesian axes, but these displacements were not examined in detail. If the displacement at the mid-surface of the element is considered, they can be written in terms of the nodal displacements and shape functions, in a similar way to the global coordinates in equation (137), thus,

$$u_m = \sum_{i=1}^{n} N_i u_i$$

$$v_m = \sum_{i=1}^{n} N_i v_i \qquad (159)$$

$$z_m = \sum_{i=1}^{n} N_i w_i$$

The displacement components at any point through the thickness of the shell must be determined. It was shown in equation (148) that the component of the global coordinate through the thickness of the shell was a function of the direction cosines of the local z_l axis, the intrinsic coordinate ζ and the thickness of the shell. It would be reasonable to assume that these will affect the displacement at any point through the thickness of the shell. The rotations α and β about the local y_l and x_l will also have an effect on the displacement at any point through the thickness. *Figure 18* shows part of the displaced shell looking in the direction of the local y_l axes. The rotation about the y_l axis is α.

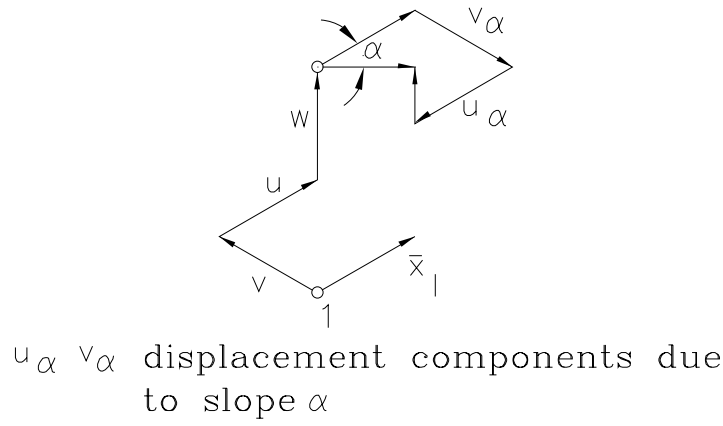

Figure 18 : Displaced shape of the shell element.

The local displacement component due to the rotation will be,

$$(u_l)_\alpha = t(\zeta - 0.5)\,\alpha \tag{160}$$

similarly the local displacement component due to the rotation β will be,

$$(v_l)_\alpha = -t(\zeta - 0.5)\,\beta \tag{161}$$

The global displacement components due to the rotation can be found using the transformation matrix of direction cosines.

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_{rotation} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{Bmatrix} u_l \\ v_l \\ 0 \end{Bmatrix}_{rotation} \tag{162}$$

The rotation at any point in the element can be specified by the nodal rotations and the Lagrangian shape functions as,

$$\begin{aligned} \alpha &= \sum_{i=1}^{n} N_i\,\alpha_i \\ \beta &= \sum_{i=1}^{n} N_i \beta_i \end{aligned} \tag{163}$$

Using equations (159), (162) and (163) the global displacement components can therefore, be written in full as,

$$u = \sum_{i=1}^{n} N_i u_i + \sum_{i=1}^{n} N_i t(\zeta - 0.5)(l_1 \alpha_i - l_2 \beta_i)$$

$$v = \sum_{i=1}^{n} N_i v_i + \sum_{i=1}^{n} N_i t(\zeta - 0.5)(m_1 \alpha_i - m_2 \beta_i) \qquad (164)$$

$$z = \sum_{i=1}^{n} N_i w_i + \sum_{i=1}^{n} N_i t(\zeta - 0.5)(n_1 \alpha_i - n_2 \beta_i)$$

Now that the displacement have been established the strain-displacement relationship can be investigated. The strain-displacement relationships can be written as,

$$\epsilon_x = \frac{\partial u}{\partial x}$$

$$\epsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad (165)$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

As the displacements functions have been defined in terms of the intrinsic coordinate system, it would be useful if the strains were also specified in the same system. This would make evaluation simpler. Consider the three displacement components differentiated with respect to the three intrinsic coordinate axes. Thus,

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \xi} \qquad (166)$$

In total nine such expressions can be written and these can be expressed in matrix form as,

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial v}{\partial \xi} & \frac{\partial w}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} & \frac{\partial w}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} & \frac{\partial v}{\partial \zeta} & \frac{\partial w}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \zeta} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \qquad (167)$$

It can be seen, that in equation (167), the left-hand matrix contains all the components of strain. This matrix is multiplied by the Jacobian matrix. Equation (167) can be re-arranged to give the strain components in terms of the intrinsic parameters.

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial v}{\partial \xi} & \frac{\partial w}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} & \frac{\partial w}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} & \frac{\partial v}{\partial \zeta} & \frac{\partial w}{\partial \zeta} \end{bmatrix} \tag{168}$$

The strains can be written in terms of the local coordinates using the standard operation,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{global} = [T] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{local} [T]^T \tag{169}$$

Noting that the transpose of the transformation matrix is equal its inverse, the equation (168) can be written as,

$$\begin{bmatrix} \frac{\partial u_l}{\partial x_l} & \frac{\partial v_l}{\partial x_l} & \frac{\partial w_l}{\partial x_l} \\ \frac{\partial u_l}{\partial y_l} & \frac{\partial v_l}{\partial y_l} & \frac{\partial w_l}{\partial y_l} \\ \frac{\partial u_l}{\partial z_l} & \frac{\partial v_l}{\partial z_l} & \frac{\partial w_l}{\partial z_l} \end{bmatrix} = \left([J][T]^T\right)^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial v}{\partial \xi} & \frac{\partial w}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} & \frac{\partial w}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} & \frac{\partial v}{\partial \zeta} & \frac{\partial w}{\partial \zeta} \end{bmatrix} [T]^T \tag{170}$$

Defining,

$$[A] = [J][T]^T \tag{171}$$

The matrix [A] can be written as,

$$[A] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \zeta} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \tag{172}$$

Thus,

$$\begin{matrix} {}_1\dfrac{\partial x}{\partial \xi} + m_1\dfrac{\partial y}{\partial \xi} + n_1\dfrac{\partial z}{\partial \xi} & l_2\dfrac{\partial x}{\partial \xi} + m_2\dfrac{\partial y}{\partial \xi} + n_2\dfrac{\partial z}{\partial \xi} & l_3\dfrac{\partial x}{\partial \xi} + m_3\dfrac{\partial y}{\partial \xi} \\ {}_1\dfrac{\partial x}{\partial \eta} + m_1\dfrac{\partial y}{\partial \eta} + n_1\dfrac{\partial z}{\partial \eta} & l_2\dfrac{\partial x}{\partial \eta} + m_2\dfrac{\partial y}{\partial \eta} + n_2\dfrac{\partial z}{\partial \eta} & l_3\dfrac{\partial x}{\partial \eta} + m_3\dfrac{\partial y}{\partial \mathrm{r}} \\ {}_1\dfrac{\partial x}{\partial \zeta} + m_1\dfrac{\partial y}{\partial \zeta} + n_1\dfrac{\partial z}{\partial \zeta} & l_2\dfrac{\partial x}{\partial \zeta} + m_2\dfrac{\partial y}{\partial \zeta} + n_2\dfrac{\partial z}{\partial \zeta} & l_3\dfrac{\partial x}{\partial \zeta} + m_3\dfrac{\partial y}{\partial \zeta} \end{matrix} \tag{173}$$

Some of the components of the matrix [A] can be evaluated. Using the definition of the direction cosines of the z_l axis, given in equation (144), it can be shown that,

$$A_{13} = A_{23} = 0 \tag{174}$$

Using the values of the global coordinates given in equation (148), it can be seen that,

$$\begin{aligned} \frac{\partial x}{\partial \zeta} &= l_3 t \\ \frac{\partial y}{\partial \zeta} &= m_3 t \\ \frac{\partial z}{\partial \zeta} &= n_3 t \end{aligned} \tag{175}$$

Substituting these values into the components A_{31} and A_{32} of the [A] matrix gives,

$$\begin{aligned} A_{31} &= l_1 l_3 + m_1 m_3 + n_1 n_3 \\ A_{32} &= l_2 l_3 + m_2 m_3 + n_2 n_3 \end{aligned} \tag{176}$$

As the three local axis are mutually perpendicular both of the expressions result in zero values. Therefore the matrix [A] can be written as,

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \tag{177}$$

where,

$$
\begin{aligned}
A_{11} &= l_1\frac{\partial x}{\partial \xi} + m_1\frac{\partial y}{\partial \xi} + n_1\frac{\partial z}{\partial \zeta} \\
A_{12} &= l_2\frac{\partial x}{\partial \xi} + m_2\frac{\partial y}{\partial \xi} + n_2\frac{\partial z}{\partial \xi} \\
A_{21} &= l_1\frac{\partial x}{\partial \eta} + m_1\frac{\partial y}{\partial \eta} + n_1\frac{\partial z}{\partial \eta} \\
A_{22} &= l_2\frac{\partial x}{\partial \eta} + m_2\frac{\partial y}{\partial \eta} + n_2\frac{\partial z}{\partial \eta} \\
A_{33} &= l_3\frac{\partial x}{\partial \zeta} + m_3\frac{\partial y}{\partial \zeta} + n_3\frac{\partial z}{\partial \zeta}
\end{aligned}
\tag{178}
$$

To evaluate equation (170), the inverse of the [A] matrix must be found. It can be shown that,

$$
= \frac{1}{A_{11}A_{22}A_{33} - A_{12}A_{21}A_{33}}
\begin{bmatrix}
A_{22}A_{33} & -A_{21}A_{33} & 0 \\
-A_{21}A_{33} & A_{11}A_{33} & 0 \\
0 & 0 & A_{11}A_{22} - A
\end{bmatrix}
\tag{179}
$$

For ease of derivation, the following parameter is defined,

$$
[C] = [A]^{-1} \tag{180}
$$

The local strain tensor can now be evaluated from the expression,

$$
[\epsilon]_{local} = [C]
\begin{bmatrix}
\frac{\partial u}{\partial \xi} & \frac{\partial v}{\partial \xi} & \frac{\partial w}{\partial \xi} \\
\frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} & \frac{\partial w}{\partial \eta} \\
\frac{\partial u}{\partial \zeta} & \frac{\partial v}{\partial \zeta} & \frac{\partial w}{\partial \zeta}
\end{bmatrix}
\begin{bmatrix}
l_1 & l_2 & l_3 \\
m_1 & m_2 & m_3 \\
n_1 & n_2 & n_3
\end{bmatrix}
\tag{181}
$$

Considering the first component of strain, it can be shown that,

$$
= C_{11}\left(l_1\frac{\partial u}{\partial \xi} + m_1\frac{\partial v}{\partial \xi} + n_1\frac{\partial w}{\partial \xi} \right) + C_{12}\left(l_1\frac{\partial u}{\partial \eta} + m_1\frac{\partial v}{\partial \eta} + \frac{\partial}{\partial} \right) \tag{182}
$$

where,

$$
\begin{aligned}
C_{11} &= \frac{A_{22}A_{33}}{A_{11}A_{22}A_{33} - A_{12}A_{21}A_{33}} \\
C_{12} &= -\frac{A_{21}A_{33}}{A_{11}A_{22}A_{33} - A_{12}A_{21}A_{33}}
\end{aligned}
\tag{183}
$$

From equation (164) it can be shown that,

$$\frac{\partial u}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} u_i + \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} t(\zeta - 0.5)(l_1 \alpha_i - l_2 \beta_i) \tag{184}$$

similar expressions exist for $\frac{\partial v}{\partial \xi}$ and $\frac{\partial w}{\partial \xi}$. The first component of strain can be written as,

$$\begin{aligned}\frac{\partial u_l}{\partial x_l} = \sum_{i=1}^{n} \Bigg[& C_{11} \left\{ l_1 \frac{\partial N_i}{\partial \xi} u_i + m_1 \frac{\partial N_i}{\partial \xi} v_i + n_1 \frac{\partial N_i}{\partial \xi} w_i \right\} \\ & + C_{12} \left\{ l_1 \frac{\partial N_i}{\partial \eta} u_i + m_1 \frac{\partial N_i}{\partial \eta} v_i + n_1 \frac{\partial N_i}{\partial \eta} w_i \right\} \\ & + t(\zeta - 0.5) \Bigg\{ C_{11} \frac{\partial N_i}{\partial \xi} \left[(l_1 l_1 + m_1 m_1 + n_1 n_1) \alpha_i \right. \\ & \left. - (l_1 l_2 + m_1 m_2 + n_1 n_2) \beta_i \right] \\ & + C_{12} \frac{\partial N_i}{\partial \eta} \left[(l_1 l_1 + m_1 m_1 + n_1 n_1) \alpha_i \right. \\ & \left. - l_1 l_2 + m_1 m_2 + n_1 n_2) \beta_i \right] \} \Bigg]\end{aligned} \tag{185}$$

Noting that for perpendicular vectors,

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \tag{186}$$

and that,

$$l_1^2 + m_1^2 + n_1^2 = 1 \tag{187}$$

then equation (185) can be re-written in terms of vectors for the *ith* nodes as,

$$\frac{\partial u_l}{\partial x_l} = \begin{Bmatrix} l_1\left(C_{11}\frac{\partial N_i}{\partial \xi} + C_{12}\frac{\partial N_i}{\partial \eta}\right) \\ m_1\left(C_{11}\frac{\partial N_i}{\partial \xi} + C_{12}\frac{\partial N_i}{\partial \eta}\right) \\ n_1\left(C_{11}\frac{\partial N_i}{\partial \xi} + C_{12}\frac{\partial N_i}{\partial \eta}\right) \\ t(\zeta - 0.5)\left(C_{11}\frac{\partial N_i}{\partial \xi} + C_{12}\frac{\partial N_i}{\partial \eta}\right) \\ 0 \end{Bmatrix}^T \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \alpha_i \\ \beta_i \end{Bmatrix} \tag{188}$$

This procedure is carried out for the other components of the strain vector, it is found that,

$$\{\epsilon_l\} = [B]\{a\} \tag{189}$$

where for the *ith* node the [B] matrix has components,

$$\begin{matrix} l_1 G_1 & m_1 G_1 & n_1 G_1 & t(\zeta - 0.5)G_1 & \\ l_2 G_2 & m_2 G_2 & n_2 G_2 & 0 & -t(\zeta - \\ {}_1G_2 + l_2 G_1 & m_1 G_2 + m_2 G_1 & n_1 G_2 + n_2 G_1 & t(\zeta - 0.5)G_2 & -t(\zeta - \\ l_3 G_2 & m_3 G_2 & n_3 G_2 & 0 & - \\ l_3 G_1 & m_3 G_1 & n_3 G_1 & tG_3 & \end{matrix} \tag{190}$$

where,

$$\begin{aligned} G_1 &= C_{11}\frac{\partial N_i}{\partial \xi} + C_{12}\frac{\partial N_i}{\partial \eta} \\ G_2 &= C_{21}\frac{\partial N_i}{\partial \xi} + C_{22}\frac{\partial N_i}{\partial \eta} \\ G_3 &= C_{33}N_i \end{aligned} \tag{191}$$

Thus the local strains have been defined in terms of the global displacement components. Equation (132) gives the expression for the strain energy. The strain energy of the element will have the same value whether the global or local stresses are used. It has also been seen that the local stress is related to the local strain by the [D] matrix. Thus the strain energy expression can be written as,

$$U = \frac{1}{2}\{a\}^T\left[\int_{Vol}[B]^T[D][B]\,dVol\right]\{a\} \tag{192}$$

Thus the total potential energy can be written as,

$$\Pi = \frac{1}{2}\{a\}^T\left[\int_{Vol}[B]^T[D][B]\,dVol\right]\{a\} - \{a\}^T\{f\} \tag{193}$$

This expression is minimised with respect to the nodal displacements to give the finite element equation,

$$[K]\{a\} = \{f\} \tag{194}$$

where [K] is the element stiffness matrix.

The element stiffness matrix can be partially evaluated analytically, as the limits of integration in the direction of the *z* axis, will always be between,

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} dz = t \tag{195}$$

The limits of integration can be changed from cartesian coordinates to the intrinsic coordinates using the standard method,

$$\int_x\int_y dy\,dx = \int_\xi\int_\eta |J|\,d\eta\,d\xi \tag{196}$$

One final modification is made to the equation for the stiffness matrix, this is to modify the [D] matrix with a correction factor for the shear deformation terms. The [D] matrix becomes,

$$[D] = \frac{E}{(1-nu^2)}\begin{bmatrix} 1 & nu & 0 & 0 & 0 \\ nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-nu}{2\alpha} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-nu}{2\alpha} \end{bmatrix} \tag{197}$$

where $\alpha = 1.2$ for isotropic materials.

The actual shear distribution through the thickness of the plate is parabolic, but the distribution obtained from the displacement definition will be constant. The correction factor is used to ensure that the strain energy due to the shear is approximately equal to that of the actual distribution.

6.2 Semi-Loof Elements

The semi-loof element is one of the most popular shell element found in todays finite element packages. The element has general applicability and produces accurate results. It is extremely difficult to formulate and is rarely understood by engineers, but is has been shown to be an extremely useful analysis tool. Due to the complex nature of this element, this section only gives outline information to the semi-loof element.

The element was developed by Bruce Irons because of a deep dissatisfaction with the existing available thin shell elements at that time. The starting point was to outline what was required from the element. This took into consideration the both the user and the programmer. The points of what were considered to be a *good* element are listed below,

1. Finite element packages have a general applicability and the element should be a direct stiffness element, suitable for such a package.

2. The shell element should be able to be attached to other elements to model practical situations. (There is a semi-loof beam element that can be used as a stiffener for semi-loof shell elements and the degrees of freedom at the nodes match for both elements).

3. The element must be able to analyse shells of small thickness.

4. The element must be able to pass the patch test (also developed by Bruce Irons). To achieve this the element must allow strain-free rigid body motion.

5. The element should be able to deal with sharp corners, junctions and other practical geometrical features without complications.

6. The element should be robust in performance.

7. Extreme accuracy is never needed (How accurate are the material properties used for the model?), however, a course mesh should still produce reasonable results.

8. The element should be implemented as a shape function routine, which encapsulates all the complications.

The numerous shell elements available before the development of the semi-loof satisfied some of these criteria but not them all.

The semi-Loof shell is based on the Kirchhoff theory (the basic assumptions of which are given in Section *.*) and is essentially for thin plates. The element is available as a six node triangle and eight node quadrilateral. The element also has nine Loof points (named after H. W. Loof) which are situated at the integration points along the sides of the element and at the centre of the element. These points are transparent to the user. The Loof points are at the points of a 2x2 integration scheme, but on the edge of the element. At each node there are three degrees of freedom (*u*, *v*, *w*), translations in the global *x*, *y* and *z* directions respectively. The rotations normal to an edge are the degrees of freedom at the Loof points, but these are generally combined and given as a single degree of freedom at the mid-side nodes. The element degrees of freedom are shown below in *Figure 19*.

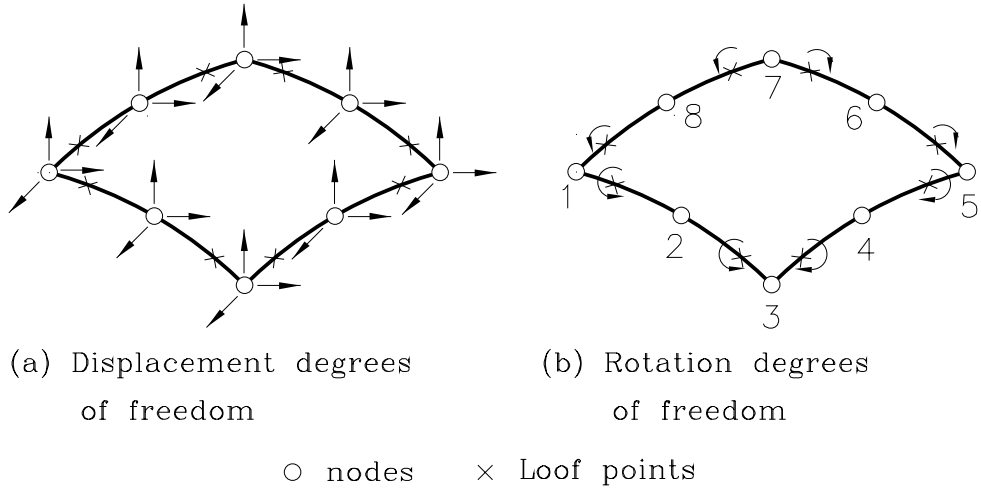

Figure 19 : The semi-loof element with degrees of freedom.

The element is a non-conforming element, but the normal rotations at the Loof nodes provide some measure of C^1 continuity. The starting point of the derivation was a forty five degree of freedom element, the degrees of freedom being three displacement components at each of the eight nodes and the centre node, a normal and tangential rotation at each of the Loof points and two rotations about the local axes of the central node. Using a series of constraint relationships the number of degrees of freedom are reduced firstly to forty three and finally to thirty two for the eight node quadrilateral element. The constraints come about from satisfying the patch test and using concepts of geometrical and generalized displacement definitions. The degrees of freedom are firstly reduced by two by

combining the displacement components at the central node to give the nodal deflection at the centre of the element. There are eleven other degrees of freedom removed from the element by constraining the shears at the Loof nodes to zero. This effectively removes the tangential rotations at the Loof nodes, the rotations at the centre node.

There are two sets of shape functions for the Semi-Loof element. The first are the shape functions at the nodes, which are those used for the Ahmad shell element and the two dimensional plane stress (strain) element. The second set of shape functions are those which apply to the Loof points. There are nine of these functions, eight for the points on the edge of the element and one for the Loof node at the centre of the element.

The element has a total of thirty two degrees of freedom, (considerably less than other shell elements) for the eight node element. The two normal rotation are generally combined and specified as a rotation at the mid-point nodes. The element would then appear to have twenty eight degrees of freedom. The sign convention adopted for the rotations normal to the edge is a right hand screw rule with the thumb parallel to the edge in the direction from the first specified node on the edge to the second. This can be seen in *Figure 19(b)*.

The calculation of the nodal strains presents a problem in the derivation of this element. The direct strains are obtained from the nodal shape functions. The bending and shear strains consists of a component due to the nodal displacements and the rotation at the Loof points. The Loof shape functions cannot be differentiated to give strains (as with the nodal shape functions). The bending and lateral strains are obtained by satisfying the requirements that,

- The bending behaviour of a flat plate must be reproduced
- The rigid body motions must be strain free.

The Semi-Loof element has two spurious mechanisms. That is the element deforms in such a manner that the strains are zero. This comes about by the fact that the elements deform in such a manner that the displacements at the Gauss integration points, i.e. the points where the strains are calculated, are zero. The first mechanism is the hour glass, this also occurs with plane elements. The second mechanism is *saddle* shaped. The element bends in such a manner that the Gauss points do not displace.

The semi-Loof element produces good results even with a course mesh and does not suffer when problems with sharp corners and multiple junctions occur due to the fact that there are no rotations

at the corner nodes.

7 Axisymmetric Shell Element

Much of the pioneering work carried out developing the finite element method was done by the aerospace industry. The stress integrity of the components and assemblies in this industry is of paramount concern, for safety reasons. The aircraft structures contain many types of elements, plates, spars, etc and to carry out stress analysis by hand is extremely difficult, time consuming and error prone. The aircraft must have strength so that it does not fall apart in flight, but it must also be as light as possible to maximise the carrying capacity. The finite element method is a means of accurately calculating stresses and allowing great advances in strength to weight ratios.

A form that often appears in aerospace structures is thin axisymmetric shells. The body of an aircraft, or rocket, rocket fuel tanks, rocket motor nozzles are a few examples of this type of structure. As this type of structure is so common a large amount of research was carried out to develop an axisymmetric shell element.

The axisymmetric shell element is a conical fustra, represented by a straight line element, as shown in *Figure 20*.

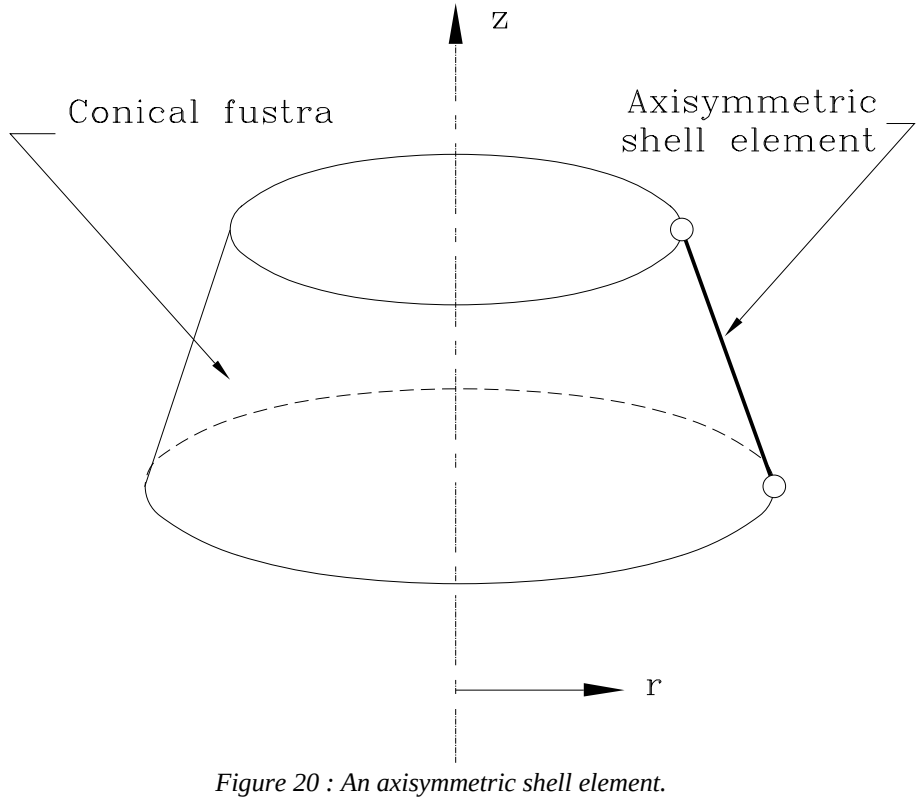

Figure 20 : An axisymmetric shell element.

A curved axisymmetric structure can be approximated using a number of sufficiently sized shell elements. The element is considered in the global *r* - *z* plane, but the element has a local coordinate system defined by the *s* - *t* axes, which is inclined at an angle ϕ, to the global coordinate system. The local coordinate system is required to define the in-plane and normal displacements, as shown in *Figure 21*.

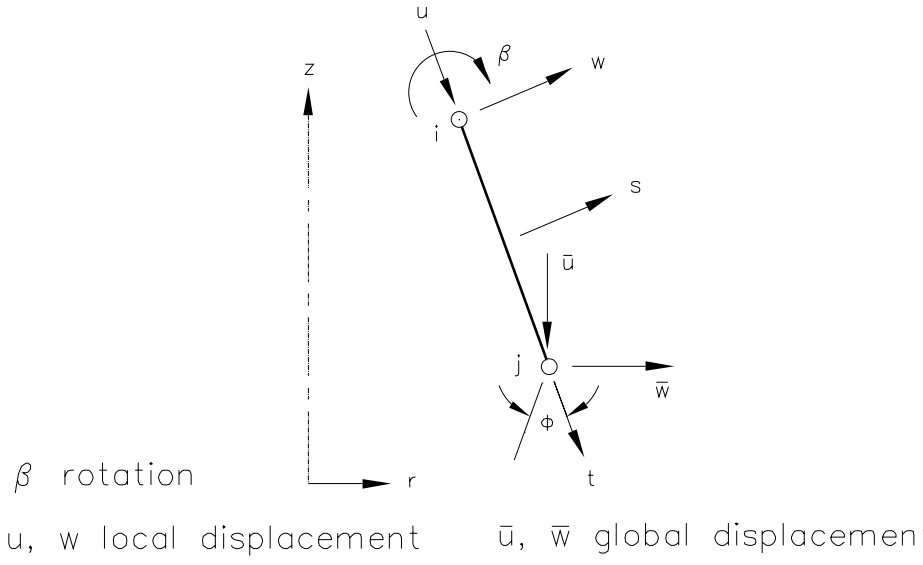

Figure 21 : Local coordinate system for axisymmetric shell element.

Each node has three degrees of freedom associated with it, these are two displacement components and a rotation. These correspond to the three local components, P_r, P_z and M.

The theory of total minimum potential energy is again used to obtain the finite element equation. The strain energy is a function of the stress and strain and as with other derivations, the generalised Hooke's law is used to obtain the strain energy in terms of strains only. The strain-displacement (***compatibility***) equations are then used to obtain the strain energy equation in terms of a 'stiffness' and displacements. This form of the strain energy equation is then used to obtain the finite element equation.

The first step will be to obtain the strain components in terms of the local displacements. There area total of four strain components, two membrane (in-plane) and two bending (out-of-plane) strains. There are direct strains and hoop strains.

If the membrane strains are considered, then it can be seen from *Figure 22*, that the strain components are,

$$\epsilon_{t_m} = \frac{du}{ds} \tag{198}$$

for the direct membrane strain and,

$$\epsilon_{\theta_m} = \frac{u\sin\phi + w\cos\phi}{r} \tag{199}$$

for the hoop strain.

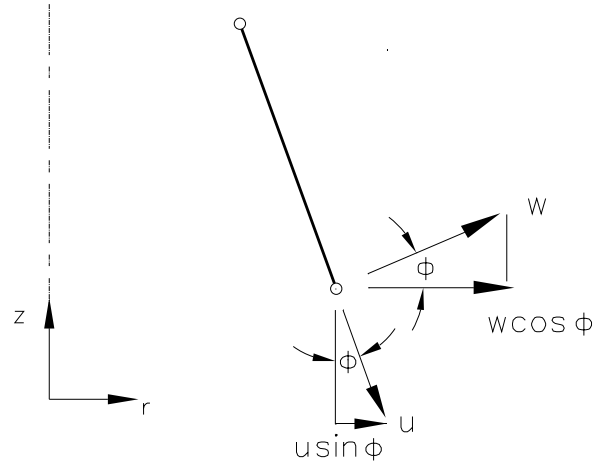

Figure 22 : In-plane strain components.

The bending strains can be deduced from *Figure 23*.

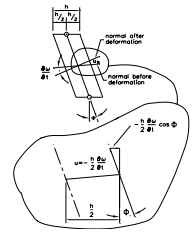

Figure 23 : Bending strain components.

Figure 23 shows the shell element and the membrane displacement due to bending. The hoop component of this displacement is shown in detail in *Figure 23*. The membrane strain due to bending is given by,

$$\epsilon_{t_b} = \frac{du}{dt} = \frac{d}{dt}\left(-\frac{h}{2}\frac{dw}{dt}\right) = -\frac{h}{2}\frac{d^2w}{dt^2} \tag{200}$$

The hoop component of strain caused by bending is given by,

$$\epsilon_{\theta_b} = -\frac{h}{2}\frac{dw}{ds}\frac{\cos\phi}{r} \tag{201}$$

The displacement vector for the element can be defined as,

$$\{u\}^t = \left[u_1 \quad w_1 \quad \frac{dw_1}{dt} \quad ..... \quad u_n \quad w_n \quad \frac{dw_n}{dt} \right] \tag{202}$$

The shape functions for the element are easily obtained from knowledge of previous derivations. As the displacement *u*, is required only and none of it's derivatives then the shape functions are Lagrangian in form and are those used for the two-noded axial bar element. This can easily be transformed to an intrinsic coordinate system, thus,

$$u = \sum_{i=1}^{n} N_i(\xi) u_i \tag{203}$$

where,

$$\begin{aligned} N_1 &= 1 - \xi \\ N_2 &= \xi \end{aligned} \tag{204}$$

and,

$$\xi = \frac{t - t_1}{L} \tag{205}$$

The displacement component *w*, is required along with it's first derivative with respect to *t*. This implies that Hermitian shape functions are required and in fact those of the two node beam element are used. The displacement can be written in terms of the shape functions and nodal displacement as,

$$w = \sum_{i=1}^{n} \left(w_i g_i + \frac{dw}{dt} h_i \right) \tag{206}$$

where,

$$\begin{aligned} g_i &= H_i^{1,0}(n, \xi) \\ h_i &= L\, H_i^{1,1}(n, \xi) \end{aligned} \tag{207}$$

where, for $H_i^{1,0}$ the subscript *i* refers to the node number and the superscript 1,0 refers to the displacement. The strain-displacement relationships can be written in matrix form as follows,

$$\begin{aligned} \{\epsilon\}_m &= [B_m]\{u\} \\ \{\epsilon\}_b &= [B_b]\{u\} \end{aligned} \tag{208}$$

where,

$$[B_m] = \begin{bmatrix} b_1 & b_2 & \ldots\ldots & b_n \end{bmatrix} \tag{209}$$

and,

$$[b_i] = \begin{bmatrix} \frac{1}{L}N_i & 0 & 0 \\ \left(\frac{\sin\phi}{r}\right)N_i & \left(\frac{\cos\phi}{r}\right)g_i & \left(\frac{\cos\phi}{r}\right)g_i \end{bmatrix} \tag{210}$$

also,

$$[B_b] = \begin{bmatrix} b_1 & b_2 & \ldots\ldots & b_n \end{bmatrix} \tag{211}$$

where,

$$\begin{bmatrix} b_i \end{bmatrix}_b = \begin{bmatrix} 0 & -\left(\frac{h}{2L^2}\right)\frac{d^2g_i}{d\xi^2} & -\left(\frac{h}{2L^2}\right)\frac{d^2h_i}{d\xi^2} \\ 0 & -\left(\frac{h\sin\phi}{2rL}\right)\frac{dg_i}{d\xi} & -\left(\frac{h\sin\phi}{2rL}\right)\frac{dh_i}{d\xi} \end{bmatrix} \tag{212}$$

Note that,

$$\frac{dw}{d\xi} = \frac{dw}{dt}\frac{dt}{d\xi} = L\frac{dw}{dt} \tag{213}$$

thus,

$$\frac{dw}{dt} = \frac{1}{L}\frac{dw}{d\xi} \tag{214}$$

and it can also be found that,

$$\frac{d^2w}{dt^2} = \frac{1}{L^2}\frac{dw}{d\xi} \tag{215}$$

There terms, which are included in equation (212) would generally form the Jacobian matrix, but due to their simple nature they are included in the [B] matrix. The stresses can be written in terms of the

strains as follows,

$$\begin{aligned} \{\sigma_m\} &= [D]\{\epsilon_m\} \\ \{\sigma_b\} &= [D]\{\epsilon_b\} \end{aligned} \tag{216}$$

where,

$$[D] = \frac{E}{1-nu^2}\begin{bmatrix} 1 & nu \\ nu & 1 \end{bmatrix} \tag{217}$$

Therefore, the strain energy for the shell element can be written in the usual format as,

$$= \{a\}^t\left[\int_{Vol}[B_m]^t[D][B_m]\,dVol + \int_{Vol}[B_b]^t[D][B_b]\,dVol\right] \tag{218}$$

This expression is differentiated with respect to the displacements and equated to zero. This gives the left hand side of the finite element equation. A membrane and bending stiffness matrix is obtained, these can be written as,

$$\begin{aligned} [K]_m &= \int_r\int_z [B_m]^t[D][B_m]\,t\,dz\,dr \\ [K]_b &= \int_r\int_z \left[\int_{-\frac{t}{2}}^{\frac{t}{2}} [B_b]^t[D][B_b]\,ds\right] dz\,dr \end{aligned} \tag{219}$$

A rotation matrix is required to convert the local coordinate system to the global coordinate system. It is found that the rotation matrix is given by,

$$[T] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{220}$$

Note the similarity between this rotation matrix and that obtained for the plane truss element. Thus it can be deduced that the global stiffness matrix can be expressed as,

$$[K]_g = [T]^t[K]_l[T] \tag{221}$$

where the subscript *g* refers to the global system and *l* to the local coordinate system.

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