Ejercicios 28 y 29 del TP1

28) Datos: $\mathbf{r}(x) = (x, f(x))$ con $x \in D(f)$ y f dos veces derivable.
Según la fórmula de cálculo se tiene que

$$\kappa(x) = \frac{|\mathbf{T}'(x)|}{|\mathbf{r}'(x)|}.$$

Por ello buscamos $|\mathbf{T}'(x)|$ y $|\mathbf{r}'(x)|$.

En primer lugar se tiene

$$\mathbf{r}'(x) = (1, f'(x)) \qquad \text{y} \qquad |\mathbf{r}'(x)| = \sqrt{1 + (f'(x))^2}.$$

Por otra parte:

$$\mathbf{T}(x) = \frac{1}{\sqrt{1 + (f'(x))^2}}(1, f'(x))$$

Aplicando la regla de derivación de escalar por vector:

$$\mathbf{T}'(x) = -\frac{1}{2}\frac{2f'(x)f''(x)}{(1 + (f'(x))^2)^{3/2}}(1, f'(x)) + \frac{1}{(1 + (f'(x))^2)^{1/2}}(0, f''(x))$$

Multiplicamos y dividimos por UNO, de manera conveniente y extraemos factor común:

$$\mathbf{T}'(x) = -\frac{1}{2}\frac{2f'(x)f''(x)}{(1 + (f'(x))^2)^{3/2}}(1, f'(x)) + \frac{1}{(1 + (f'(x))^2)^{1/2}}(0, f''(x))\frac{1 + (f'(x))^2}{(1 + (f'(x))^2)^{2/2}}$$

$$= \left(-f'(x)f''(x), -(f'(x))^2 f''(x) + f''(x)(1 + (f'(x))^2)\right)(1 + (f'(x))^2)^{-3/2}$$

$$|\mathbf{T}'(x)| = \sqrt{\frac{(f'(x))^2(f''(x))^2}{(1 + (f'(x))^2)^3} + \frac{(f''(x))^2}{(1 + (f'(x))^2)^3}}$$

$$= \sqrt{\frac{(f''(x))^2(1 + (f'(x))^2)}{(1 + (f'(x))^2)^3}} = \frac{|f''(x)|}{|1 + (f'(x))^2|}$$

$$\kappa(x) = \frac{|\mathbf{T}'(x)|}{|\mathbf{r}'(x)|} = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}} \qquad \square$$

29) Datos: $\mathbf{r}(t) = (x(t), y(t))$, todas dos veces derivables.
Según la fórmula de cálculo se tiene que

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Por ello buscamos $|\mathbf{T}'(t)|$ y $|\mathbf{r}'(t)|$.

En primer lugar se tiene

$$\mathbf{r}'(t) = (x'(t), y'(x)) = (\dot{x}, \dot{y}) \qquad \text{y} \qquad |\mathbf{r}'(t)| = \sqrt{\dot{x}^2 + \dot{y}^2}.$$

Por otra parte:

$$\mathbf{T}(t) = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}(\dot{x}, \dot{y}) = \left(\dot{x}(\dot{x}^2 + \dot{y}^2)^{-1/2}, \dot{y}(\dot{x}^2 + \dot{y}^2)^{-1/2}\right)$$

Derivando:

$$\mathbf{T}'(t) = \Big(\ddot{x}(\dot{x}^2 + \dot{y}^2)^{-1/2} + \dot{x}(-\frac{1}{2})(\dot{x}^2 + \dot{y}^2)^{-3/2}(2\dot{x}\ddot{x} + 2\dot{y}\ddot{y}),$$

$$\ddot{y}(\dot{x}^2 + \dot{y}^2)^{-1/2} + \dot{y}(-\frac{1}{2})(\dot{x}^2 + \dot{y}^2)^{-3/2}(2\dot{x}\ddot{x} + 2\dot{y}\ddot{y})\Big)$$

$$= (\dot{x}^2 + \dot{y}^2)^{-3/2}\left(\ddot{x}(\dot{x}^2 + \dot{y}^2) - \dot{x}(\dot{x}\ddot{x} + \dot{y}\ddot{y}), \ddot{y}(\dot{x}^2 + \dot{y}^2) - \dot{y}(\dot{x}\ddot{x} + \dot{y}\ddot{y}))\right)$$

$$|\mathbf{T}'(t)| = (\dot{x}^2 + \dot{y}^2)^{-3/2}\left(\left(\ddot{x}(\dot{x}^2 + \dot{y}^2) - \dot{x}(\dot{x}\ddot{x} + \dot{y}\ddot{y})\right)^2 + \left(\ddot{y}(\dot{x}^2 + \dot{y}^2) - \dot{y}(\dot{x}\ddot{x} + \dot{y}\ddot{y})\right)^2\right)^{1/2}$$

$$\kappa(t) = (\dot{x}^2 + \dot{y}^2)^{-3/2}\left(\frac{\left(\ddot{x}(\dot{x}^2 + \dot{y}^2) - \dot{x}(\dot{x}\ddot{x} + \dot{y}\ddot{y})\right)^2}{\dot{x}^2 + \dot{y}^2} + \frac{\left(\ddot{y}(\dot{x}^2 + \dot{y}^2) - \dot{y}(\dot{x}\ddot{x} + \dot{y}\ddot{y})\right)^2}{\dot{x}^2 + \dot{y}^2}\right)^{1/2}$$

$$= (\dot{x}^2 + \dot{y}^2)^{-3/2}\Big(\frac{\ddot{x}^2(\dot{x}^2 + \dot{y}^2)^2 - 2\ddot{x}\dot{x}(\dot{x}^2 + \dot{y}^2)(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + \dot{x}^2(\dot{x}\ddot{x} + \dot{y}\ddot{y})^2}{\dot{x}^2 + \dot{y}^2}$$

$$+ \frac{\ddot{y}^2(\dot{x}^2 + \dot{y}^2)^2 - 2\ddot{y}\dot{y}(\dot{x}^2 + \dot{y}^2)(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + \dot{y}^2(\dot{x}\ddot{x} + \dot{y}\ddot{y})^2}{\dot{x}^2 + \dot{y}^2}\Big)^{1/2}$$

Distribuyendo el denominador y cancelando:

$$\kappa(t) = (\dot{x}^2 + \dot{y}^2)^{-3/2}\Big(\ddot{x}^2(\dot{x}^2 + \dot{y}^2) - 2\ddot{x}\dot{x}(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + \dot{x}^2(\dot{x}\ddot{x} + \dot{y}\ddot{y})^2(\dot{x}^2 + \dot{y}^2)^{-1}$$

$$+\ddot{y}^2(\dot{x}^2 + \dot{y}^2) - 2\ddot{y}\dot{y}(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + \dot{y}^2(\dot{x}\ddot{x} + \dot{y}\ddot{y})^2(\dot{x}^2 + \dot{y}^2)^{-1}\Big)^{1/2}$$

$$\kappa(t) = (\dot{x}^2+\dot{y}^2)^{-3/2}\left(\ddot{x}^2(\dot{x}^2+\dot{y}^2) - 2\ddot{x}\dot{x}(\dot{x}\ddot{x}+\dot{y}\ddot{y}) + \ddot{y}^2(\dot{x}^2+\dot{y}^2) - 2\ddot{y}\dot{y}(\dot{x}\ddot{x}+\dot{y}\ddot{y}) + (\dot{x}\ddot{x}+\dot{y}\ddot{y})^2\right)^{1/2}$$

$$= (\dot{x}^2+\dot{y}^2)^{-3/2}\left(\ddot{x}^2\dot{x}^2 + \ddot{x}^2\dot{y}^2 - 2\ddot{x}^2\dot{x}^2 - 2\dot{x}\ddot{x}\dot{y}\ddot{y} + \ddot{y}^2\dot{x}^2 + \ddot{y}^2\dot{y}^2 - 2\dot{x}\ddot{x}\ddot{y}\dot{y} - 2\dot{y}^2\ddot{y}^2\right.$$

$$\left. + \dot{x}^2\ddot{x}^2 + 2\dot{x}\ddot{x}\ddot{y}\dot{y} + \dot{y}^2\ddot{y}^2\right)^{1/2}$$

Cancelando queda:

$$\kappa(t) = (\dot{x}^2+\dot{y}^2)^{-3/2}\left(\ddot{x}^2\dot{y}^2 - 2\dot{x}\ddot{x}\dot{y}\ddot{y} + \dot{x}^2\ddot{y}^2\right)^{1/2}$$

$$= (\dot{x}^2+\dot{y}^2)^{-3/2}\left(\ddot{x}\dot{y} - \dot{x}\ddot{y}\right)^{2/2}$$

$$= \frac{|\ddot{x}\dot{y} - \dot{x}\ddot{y}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} \qquad \square$$